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INVESTIGATION OF NUMERICAL INTEGRATION TECHNIQUES WITH APPLICATIONS TO THE NUMERICAL SOLUTION OF DIFFERENTIAL EQUATIONS

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NOMENCLATURE

$f(t)$	any function of time
$Zf(t)$	a train of impulses (equally spaced, T seconds apart) whose areas are modulated by $f(t)$ at the sampling instants
$\bar{f}(s)$	the Laplace transform of $f(t)$
$Z\bar{f}(s)$	the Laplace transform of $Zf(t)$
$\lvert E_{max} \rvert$	the largest absolute value of error in the fixed period of computation (0.8 seconds)
T	sampling interval size
Error	(exact solution - approximate solution)
Relative Error	(Error /Exact solution) multiplied by 100.0
X_n	the n^{th} term of an ordered sequence
$\doteq$	"approximate equality"

INTRODUCTION

The purpose of this report is to discuss some aspects of the approximate solution of differential equations using Newton-Cotes quadratures. The goal is to illustrate a relatively straightforward method for approximating the solution of both linear and non-linear differential equations. This method is suitable for direct application on a digital computer.

Z-transformation is introduced in the solution of differential equations as a "bookkeeping" procedure for keeping track of desired coefficients. Halijak's "Trapezoidal-Convolution" (1) and Simpson's Rule are the mechanisms for obtaining the approximate analysis of differential equations.

THE ROLE OF Z-TRANSFORMS

If ideal samplers are admitted, then Z-transformation permits an organized approximate solution of a linear system at the sampling instants. In the case of a non-linear system the solution cannot be effected solely by the algebraic manipulation of Z-transforms, but the solution can be implemented by non-linear recurrence relations. The solutions thus obtained are always approximations of the true value at the sampling instants.

If an arbitrary function of time f(t) is always multiplied by g(t) defined by

$$g(t) = \sum_{n=0}^{\infty} \delta(t-nT) \qquad (1)$$

where T is the time duration between two successive pulses, then the resulting product can be expressed in Laplace transforms as

$$L\left[g(t)\ f(t)\right] = \frac{1}{2\pi j} \oint_R \bar{f}(\lambda)\bar{g}(s-\lambda)d\lambda \tag{2}$$

where

$$\bar{g}(s) = L\left[\sum_{n=0}^{\infty} \delta(t-nT)\right] = 1+e^{-sT}+e^{-2sT}+ \dots +e^{-nsT}$$

$$= \frac{1}{1-e^{-sT}} \tag{3}$$

and R is a region in the complex λ-plane containing the poles of $\bar{f}(\lambda)$. Therefore

$$L\left[g(t)\ f(t)\right] = \frac{1}{2\pi j} \oint_R \bar{f}(\lambda)\frac{1}{1-e^{-T(s-\lambda)}}d\lambda \tag{4}$$

This particular Laplace transform is the transform of the sampled function f(t).

Equation (4) can then be written

$$Z\bar{f} = \frac{1}{2\pi j} \oint_R \bar{f}(\lambda)\frac{1}{1-e^{-T(s-\lambda)}}d\lambda \tag{5}$$

$1/(1-e^{-T(s-\lambda)})$ can be developed as a geometric progression with ratio $e^{-T(s-\lambda)}$ and an initial term of one.

Equation (5) is then written

$$Z\bar{f} = \sum_{n=0}^{\infty} \frac{1}{2\pi j} \oint \bar{f}(\lambda)e^{-nT(s-\lambda)}d\lambda \tag{6}$$

$$Z\bar{f} = \sum_{n=0}^{\infty} \frac{1}{2\pi j}e^{-nTs} \oint \bar{f}(\lambda)e^{nT\lambda}d\lambda \tag{7}$$

The integral represents the inverse Laplace transform and we obtain

$$Z\bar{f} = \sum_{n=0} e^{-nTs} f(nT) \tag{8}$$

We then define $e^{-Ts} = z$, $f(nT) \equiv f_n$ and equation (8) can be shortened to

$$Z\bar{f} = \sum_{n=0} z^n f_n \tag{9}$$

NUMERICAL APPROXIMATIONS OF THE CONVOLUTION INTEGRAL

The solution of a linear differential equation with constant coefficients can be considered as a convolution problem.

Left Riemann Sum Approximation

$$\int_0^t f(\tau)g(t-\tau)d\tau = \int_0^t g(\tau)f(t-\tau)d\tau$$

is equal to

$$\sum_{k=0}^{n-1} \int_{KT}^{(K+1)T} f(\tau)g(nT-\tau)d\tau = \sum_{k=0}^{n-1} \int_{KT}^{(K+1)T} g(\tau)f(nT-\tau)d\tau \tag{10}$$

and is approximated by

$$T\sum_{k=0}^{n-1} f_k g_{n-k} = T\sum_{k=0}^{n-1} g_k f_{n-k} \tag{11}$$

The above becomes

$$T\sum_{k=0}^{n-1} f_k g_{n-k} = T\sum_{k=0}^{n} f_k g_{n-k} - f_n g_0 \tag{12}$$

Z-transformation yields

$$Z\left[T\sum_{k=0}^{n} f_k g_{n-k} - f_n g_0\right] = T(Z\bar{f})(Z\bar{g}) - TZ\bar{f}g_0 \doteq Z\left[\overline{\overline{fg}}\right] \tag{13}$$

If we use the symmetric property of the convolution integral and take the average of the two results we find that

$$Z\left[\bar{f}\bar{g}\right] = T(Z\bar{f})(Z\bar{g}) - \frac{T}{2}(f_0 Z\bar{g} + g_0 Z\bar{f}) \qquad (14)$$

Right Riemann Sum Approximation

The right Riemann sum employs the right ordinate and yields

$$T\sum_{k=0}^{n-1} f_{k+1}\, g_{n-k-1} = T\sum_{k=0}^{n-1} g_{k+1}\, f_{n-k-1} \qquad (15)$$

$$\text{Now}\quad T\sum_{k=0}^{n-1} f_{k+1}\, g_{n-k-1} = T\sum_{k=1}^{n} f_k\, g_{n-k} = T\sum_{k=0}^{n} f_k\, g_{n-k} - f_0 g_n \qquad (16)$$

Z-transforming (16) we have

$$Z\,T\left[\sum_{k=1}^{n} f_k\, g_{n-k} - f_0 g_n\right] = T(Z\bar{f})(Z\bar{g}) - T\, f_0 Z\bar{g} \qquad (17)$$

If we use the symmetric property of the convolution integral and take the average of the two results we find that

$$Z\left[\bar{f}\bar{g}\right] = T(Z\bar{f})(Z\bar{g}) - \frac{T}{2}(f_0 Z\bar{g} + g_0 Z\bar{f}) \qquad (18)$$

Trapezoidal Convolution

By definition, the area under the curve using trapezoids is

$$J_n = \frac{T}{2}\sum_{k=0}^{n-1} f_k\, g_{n-k} + f_{k+1}\, g_{n-k-1} \qquad (19)$$

$$J_n = \frac{T}{2}\sum_{k=0}^{n} f_k\, g_{n-k} - f_n g_0 + \frac{T}{2}\sum_{k=1}^{n} f_k\, g_{n-k} \qquad (20)$$

$$J_n = \frac{T}{2}\sum_{k=0}^{n} f_k\, g_{n-k} - f_n g_0 + \frac{T}{2}\sum_{k=0}^{n} f_k\, g_{n-k} - f_0 g_n \qquad (21)$$

Z-transformation yields

$$Z\left[\overline{\overline{fg}}\right] = \frac{T}{2}(Z\bar{\bar{f}})(Z\bar{\bar{g}}) - \frac{T}{2}Z\bar{\bar{f}} + \frac{T}{2}(Z\bar{\bar{f}})(Z\bar{\bar{g}}) - \frac{T}{2}f_0 Z\bar{\bar{g}}$$

and,

$$Z\left[\overline{\overline{fg}}\right] = T(Z\bar{\bar{f}})(Z\bar{\bar{g}}) - \frac{T}{2}(Z\bar{\bar{f}}g_0 + Z\bar{\bar{g}}f_0) \qquad (22)$$

It is observed that equations (14), (18), and (22) are all the same. Hence by using the symmetric property of the convolution integral the equation of trapezoidal convolution, (22), may be arrived at by using left or right hand Riemann sums.

Modified Simpson's Convolution

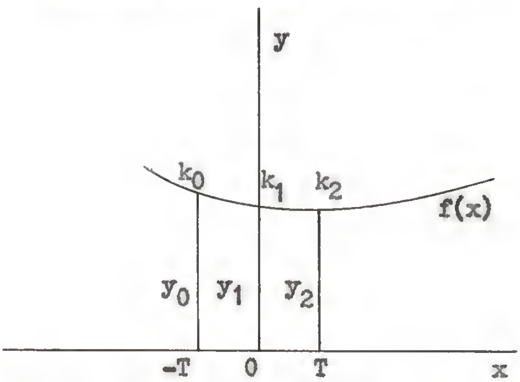

Fig. 1. Curve used for Simpson's rule.

In Fig. 1 we know the area between the ordinates y_0 and y_2 is,

$$\gamma_2 = \frac{T}{3}(y_0 + 4y_1 + y_2) \qquad (23)$$

where γ_i is the area between the i^{th} ordinate and the y_0 ordinate, and γ_{ij} is the area between the i^{th} and j^{th} ordinates.

Simpson's rule depends on an even number of intervals; this is one aspect of Simpson's rule which must be modified. It will be required that only one interval be used in an approximation. Therefore we proceed as follows: Simpson's rule requires three points to fit the parabola $aX^2 + bX + c$. From Fig. 1

$$\gamma_1 = \int_{-T}^{0} (aX^2+bX+c)dX = \frac{aX^3}{3} + \frac{bX^2}{2} + cX \Big|_{-T}^{0} = \frac{aT^3}{3} - \frac{bT^2}{2} + cT \qquad (24)$$

at $X = -T$ $\qquad y_0 = aT^2 - bT + c$

at $X = 0$ $\qquad y_1 = c$

at $X = T$ $\qquad y_2 = aT^2 + bT + c$

The next step is to find some combination of y_0, y_1, and y_2 to yield γ_1. The first observation is to multiply this particular combination by T to obtain the desired aT^3, bT^2, and cT. Therefore, equating γ_1 and the sum of y_0, y_1, and y_2 we have

$$T(By_0 + Cg_1 + Dg_2) = \frac{aT^3}{3} - \frac{bT^2}{2} + cT \qquad (25)$$

where B, C, and D represent the desired coefficients.

Therefore, we have

$$T(BaT^2 - BbT + Bc + Cc + DaT^2 + DbT + Dc) = \frac{aT^3}{3} - \frac{bT^2}{2} + cT \qquad (26)$$

Now equating coefficients of aT^3, bT^2, cT;

$(B+D) = 1/3$, $(-B+D) = -1/2$, $B+C+D = 1$

Solving these equations we find that $B = 5/12$, $D = -1/12$ and $C = 2/3$.

From this

$$\gamma_1 = (\frac{5}{12}y_0 + \frac{2}{3}y_1 - \frac{1}{12}y_2)T \qquad (27)$$

Next we find γ_{12}: We know that

$$\gamma_2 = \frac{T}{3}y_0 + \frac{4}{3}Ty_1 + \frac{T}{3}y_2$$

and

$$\gamma_{12} = \gamma_2 - \gamma_1 = \frac{1}{3}Ty_0 + \frac{4}{3}Ty_1 + \frac{1}{3}Ty_2 - \frac{5}{12}Ty_0 - \frac{2}{3}Ty_1 + \frac{1}{12}Ty_2$$

$$\gamma_{12} = \left(-\frac{1}{12}y_0 + \frac{2}{3}y_1 + \frac{5}{12}y_2\right)T \tag{28}$$

If we calculate γ_{23} the same way as γ_1, we get

$$\gamma_{23} = \left(-\frac{1}{12}y_1 + \frac{2}{3}y_2 + \frac{5}{12}y_3\right)T \tag{29}$$

Also,

$$\gamma_{34} = \left(-\frac{1}{12}y_2 + \frac{2}{3}y_3 + \frac{5}{12}y_4\right)T \tag{30}$$

From (28), (29), and (30) it is observed that a pattern exists resulting in the following recurrence relation

$$\gamma_{(n-1),n} = \left(-\frac{1}{12}y_{n-2} + \frac{2}{3}y_{n-1} + \frac{5}{12}y_n\right)T \tag{31}$$

$$n = 2, 3, 4, 5, 6, \ldots$$

From (31) we can establish the relation

$$\gamma_n = \gamma_{n-1} + \left(-\frac{1}{12}y_{n-2} + \frac{2}{3}y_{n-1} + \frac{5}{12}y_n\right)T \tag{32}$$

$$n = 2, 3, 4, 5, 6, \ldots$$

The preceding development now enables us to make an approximation to the convolution integral,

$$\int_0^{nT} f(nT-\tau)\, g(\tau)d\tau \tag{33}$$

From (33), $f(nT-\tau)\ g(\tau)$, represents the ordinate values of y_0, y_1, y_2, y_3, ... etc. for various values of n. If n=1 we have from equation (33)

$$\int_0^T f(T-\tau)\ g(\tau)d\tau$$

$$T = 0; \qquad y_0 = f_1 g_0$$

$$T = 1; \qquad y_1 = f_0 g_1$$

$$T = 2; \qquad y_2 = f_{-1} g_2$$

Normally f_{-1} would be taken to be zero, but in this case the extra point is needed to determine the direction of the curve, since Simpson's rule is based on a curve passing through 3 points.

Next it is known that

$$\gamma_1 = (\frac{5}{12}y_0 + \frac{2}{3}y_1 - \frac{1}{12}y_2)T$$

therefore

$$\gamma_1 = (\frac{5}{12}f_1 g_0 + \frac{2}{3}f_0 g_1 - \frac{1}{12}f_{-1} g_2)T \tag{34}$$

When n=2 we have from equation (33)

$$\int_0^{2T} f(2T-\tau)\ g(\tau)d\tau$$

$$T = 0; \qquad y_0 = f_2 g_0$$

$$T = 1; \qquad y_1 = f_1 g_1$$

$$T = 2; \qquad y_2 = f_0 g_2$$

and we know

$$\gamma_2 = \frac{T}{3}(y_0 + 4y_1 + y_2)$$

therefore

$$\gamma_2 = \frac{T}{3}(f_2 g_0 + 4f_1 g_1 + f_0 g_2) \tag{35}$$

When n=3 we have from equation (33)

$$\int_0^{3T} f(3T-\tau)\ g(\tau)d\tau$$

$T = 0;\qquad y_0 = f_3g_0$

$T = 1;\qquad y_1 = f_2g_1$

$T = 2;\qquad y_2 = f_1g_2$

From equation (32),

$$\gamma_3 = \gamma_2 +(-\frac{1}{12}y_1 + \frac{2}{3}y_2 + \frac{5}{12}y_3)T$$

$$= \frac{T}{3}(y_0 + 4y_1 + y_2) + (-\frac{1}{12}y_1 + \frac{2}{3}y_2 + \frac{5}{12}y_3)T$$

$$= \frac{T}{3}(f_3g_0 + 4f_2g_1 + f_1g_2) + (-\frac{1}{12}f_2g_1 + \frac{2}{3}f_1g_2 + \frac{5}{12}f_0g_3)T \quad (36)$$

When n=4 we have from equation (33)

$$\int_0^{4T} f(4T-\tau)\ g(\tau)d\tau$$

$T = 0;\qquad y_0 = f_4g_0$

$T = 1;\qquad y_1 = f_3g_1$

$T = 2;\qquad y_2 = f_2g_2$

$T = 3;\qquad y_3 = f_1g_3$

$T = 4;\qquad y_4 = f_0g_4$

From equation (32)

$$\gamma_4 = \gamma_3 + (-\frac{1}{12}y_2 + \frac{2}{3}y_3 + \frac{5}{12}y_4)T$$

$$= T(\frac{1}{3}y_0 + \frac{5}{4}y_1 + y_2 + \frac{5}{12}y_3) + (-\frac{1}{12}y_2 + \frac{2}{3}y_3 + \frac{5}{12}y_4)T$$

$$= T(\frac{1}{3}f_4g_0 + \frac{5}{4}f_3g_1 + f_2g_2 + \frac{5}{12}f_1g_3) + (-\frac{1}{12}f_2g_2 + \frac{2}{3}f_1g_3 + \frac{5}{12}f_0g_4)T$$

and

$$\gamma_4 = T(\tfrac{1}{3}f_4g_0 + \tfrac{5}{4}f_3g_1 + \tfrac{11}{12}f_2g_0 + \tfrac{13}{12}f_1g_3 + \tfrac{5}{12}f_0g_4) \tag{37}$$

In a like manner,

$$\gamma_5 = T(\tfrac{1}{3}f_5g_0 + \tfrac{5}{4}f_4g_4 + \tfrac{11}{12}f_3g_2 + f_2g_3 + \tfrac{13}{12}f_1g_1 + \tfrac{5}{12}f_0g_5) \tag{38}$$

and

$$\gamma_6 = T(\tfrac{1}{3}f_6g_0 + \tfrac{5}{4}f_5g_1 + \tfrac{11}{12}f_4g_2 + f_3g_3 + f_2g_4 + \tfrac{13}{12}f_1g_5 + \tfrac{5}{12}f_0g_6) \tag{39}$$

At this point it is observed that a definite trend is being followed in that the first three coefficients and the last two coefficients of $\gamma_4, \gamma_5, \gamma_6, \ldots \gamma_n$, are remaining constant. This is the fact that allows a closed form to the approximation of the convolution integral.

Next, write each area in the form

$$\gamma_0 = 0$$

$$\gamma_1 = \frac{5}{12}Tf_1g_0 + \frac{2}{3}Tf_0g_1 - \frac{1}{12}Tf_{-1}g_2 \tag{40}$$

$$\gamma_2 = \frac{1}{3}Tf_2g_0 + \frac{4}{3}Tf_1g_1 + \frac{1}{3}Tf_0g_2 \tag{41}$$

$$\gamma_3 = \frac{1}{3}Tf_3g_0 + \frac{5}{4}Tf_2g_1 + Tf_1g_2 + \frac{5}{12}Tf_0g_3 \tag{42}$$

$$\gamma_4 = \frac{1}{3}Tf_4g_0 + \frac{5}{4}Tf_3g_1 + \frac{11}{12}Tf_2g_2 + \frac{13}{12}Tf_1g_3 + \frac{5}{12}Tf_0g_4 \tag{43}$$

$$\gamma_5 = \frac{1}{3}Tf_5g_0 + \frac{5}{4}Tf_4g_1 + \frac{11}{12}Tf_3g_2 + Tf_2g_3 + \frac{13}{12}Tf_1g_4 + \frac{5}{12}Tf_0g_5 \tag{44}$$

$$\gamma_6 = \frac{1}{3}Tf_6g_0 + \frac{5}{4}Tf_5g_1 + \frac{11}{12}Tf_4g_2 + Tf_3g_3 + Tf_2g_4 + \frac{13}{12}Tf_1g_5 + \frac{5}{12}Tf_0g_6 \tag{45}$$

It should be noted at this time that the subscripts of each ordinate value add arithmetically to n, i.e., from (41), f_2g_0, f_1g_1, f_0g_2, we have each ordinate subscript adding to 2 which is the subscript of the corresponding γ. It is also observed that each γ_n has n+1 terms and that all terms added for each new γ_n, (for γ_4, γ_5, γ_6, ...), besides the first three and last two terms, have a constant coefficient of 1. One more item that should be taken into account is the fact that in each γ_n, ($n = 2, 3, 4, 5, 6, \ldots$), the following sequence of ordinates occur; for γ_4, we have, f_4g_0, f_3g_1, f_2g_2, f_1g_3, f_0g_4; i.e., the subscript on f starts at n and decreases to f_0 in n+1 terms and the subscripts on g do just the opposite.

Now

$$(Z\bar{f})(Z\bar{g}) = \sum_{k=0}^{n} f_k\, g_{n-k}$$

$$= f_0g_0z^0 + (f_1g_0 + f_0g_1)z + (f_2g_0 + f_1g_1 + f_0g_2)z^2 + \ldots \tag{46}$$

Comparing (46) with $\gamma_0 = 0$ and equations (40) through (45) we see that γ_0 corresponds to the z^0 term of $(Z\bar{f})(Z\bar{g})$, γ_1 corresponds to the z^1 term, γ_2 to the z^2 term, etc.

In this next step T will be suppressed from equations (40) through (45) and the necessary corrections to $(Z\bar{f})(Z\bar{g})$ will be made to account for coefficients other than 1, or any extra or deleted terms that need taken care of in equations (40) through (45).

Comparing (46) with equations (40) through (45) the following corrections are applied to $(Z\bar{f})(Z\bar{g})$;

$$(Z\bar{f})(Z\bar{g}) - \frac{2}{3}g_0Z\bar{f} + \frac{1}{4}g_1zZ\bar{f} - \frac{1}{12}g_2z^2Z\bar{f} + \frac{1}{12}f_1zZ\bar{g} - \frac{7}{12}f_0Z\bar{g}$$

Collecting terms we have:

$$(Z\bar{f})(Z\bar{g}) = \left(\frac{2}{3}g_0 - \frac{1}{4}g_1 z + \frac{1}{12}g_2 z^2\right)Z\bar{f} - \left(-\frac{1}{12}f_1 z + \frac{7}{12}f_0\right)Z\bar{g} \qquad (47)$$

Equation (47) is a corrected equation for the first three terms and last two terms mentioned on page 11.

Multiplying the last two terms of (47) out

$$\begin{array}{llllllllll}
f_0 & f_1 & f_2 & f_3 & f_4 & f_5 & & & \\
\frac{2}{3}g_0 & -\frac{1}{4}g_1 & +\frac{1}{12}g_2 & & & & & & \\
\hline
\frac{2}{3}f_0g_0 & \frac{2}{3}f_1g_0 & \frac{2}{3}f_2g_0 & \frac{2}{3}f_3g_0 & \frac{2}{3}f_4g_0 & \frac{2}{3}f_5g_0 & & \\
 & -\frac{1}{4}f_0g_1 & -\frac{1}{4}f_1g_1 & -\frac{1}{4}f_2g_1 & -\frac{1}{4}f_3g_1 & -\frac{1}{4}f_4g_1 & -\frac{1}{4}f_5g_1 & \\
 & & \frac{1}{12}f_0g_2 & \frac{1}{12}f_1g_2 & \frac{1}{12}f_2g_2 & \frac{1}{12}f_3g_2 & \frac{1}{12}f_4g_2 & \frac{1}{12}f_5g_2 \\
\hline
\end{array}$$

and

$$\begin{array}{lllllll}
g_0 & g_1 & g_2 & g_3 & g_4 & g_5 & \\
\frac{7}{12}f_0 & -\frac{1}{12}f_1 & & & & & \\
\hline
\frac{7}{12}f_0g_0 & \frac{7}{12}f_0g_1 & \frac{7}{12}f_0g_2 & \frac{7}{12}f_0g_3 & \frac{7}{12}f_0g_4 & \frac{7}{12}f_0g_5 & \\
 & -\frac{1}{12}f_1g_0 & -\frac{1}{12}f_1g_1 & -\frac{1}{12}f_1g_2 & -\frac{1}{12}f_1g_3 & -\frac{1}{12}f_1g_4 & -\frac{1}{12}f_1g_5 \\
\hline
\end{array}$$

Adding coefficients of these two terms and inserting a minus sign we have

z^0	z^1	z^2	z^3	z^4	z^5
$-\frac{15}{12}f_0g_0$	$-\frac{1}{3}f_0g_1$	$-\frac{2}{3}f_2g_0$	$-\frac{2}{3}f_3g_0$	$-\frac{2}{3}f_4g_0$	$-\frac{2}{3}f_5g_0$
	$-\frac{7}{12}f_1g_0$	$\frac{1}{3}f_1g_1$	$\frac{1}{4}f_2g_1$	$\frac{1}{4}f_3g_1$	$\frac{1}{4}f_4g_1$
		$-\frac{2}{3}f_0g_2$	$-\frac{7}{12}f_0g_3$	$-\frac{1}{12}f_2g_2$	$-\frac{1}{12}f_3g_2$
				$-\frac{7}{12}f_0g_4$	$-\frac{7}{12}f_0g_5$
				$\frac{1}{12}f_1g_3$	$\frac{1}{12}f_1g_4$

Adding these coefficients to the coefficients of $(Z\bar{f})(Z\bar{g})$ we get

$$-\frac{1}{4}f_0g_0 + \left(\frac{2}{3}f_0g_1 + \frac{5}{12}f_1g_0\right)z + \left(\frac{1}{3}f_2g_0 + \frac{4}{3}f_1g_1 + \frac{1}{3}f_0g_2\right)z^2$$

$$+ \left(\frac{1}{3}f_3g_0 + \frac{5}{4}f_2g_1 + \frac{5}{12}f_0g_3\right)z^3 + \left(\frac{1}{3}f_4g_0 + \frac{5}{4}f_3g_1 + \frac{11}{12}f_2g_2 + \frac{5}{12}f_0g_4\right.$$

$$\left. + \frac{13}{12}f_1g_3\right)z^4 + \left(\frac{1}{3}f_5g_0 + \frac{5}{4}f_4g_1 + \frac{11}{12}f_3g_2 + \frac{5}{12}f_0g_5 + \frac{13}{12}f_1g_4 + f_2g_3\right)z^5 \tag{48}$$

Comparing these results with $\gamma_0 = 0$ and equations (40) through (45), the following is concluded:

$\gamma_0 = 0$, therefore we will need to add $\frac{1}{4}f_0g_0$.

For γ_1, we need to add $-\frac{1}{12}f_{-1}g_2z$

The rest of the γ's correspond exactly and no further correction terms are required.

Therefore, we may write down the following approximation to the convolution integral.

$$Z\left[\overline{fg}\right] = T(Z\bar{f})(Z\bar{g}) - T\left(\frac{2}{3}g_0 - \frac{1}{4}g_1 z + \frac{1}{12}g_2 z^2\right)Z\bar{f} - T\left(\frac{7}{12}f_0 - \frac{1}{12}f_1 z\right)Z\bar{g}$$

$$+ \frac{T}{4}f_0 g_0 - \frac{T}{12}f_{-1} g_2 z$$

This formula will be called, "<u>The Modified Simpson's Convolution Formula</u>".

$Z\left[1/s^n\right]$ FOR THE INTEGRATOR SUBSTITUTION PROGRAM

Halijak has developed an approximate Z-transform of $\left[1/s^n\right]$ from the trapezoidal convolution formula (1). For n = 1, 2, 3 this approximation coincides with the exact solution.

Approximation of $Z\left[1/s^n\right]$ by the Modified Simpson's Convolution Formula

In a similar manner, as that in (1), we find equation (49) yields

$$Z\left[1/s^{n+1}\right] = \alpha^{n-1}\frac{Tz}{(1-z)^2}\left(\frac{7}{6} - \frac{1}{3}z + \frac{1}{6}z^2\right) + \sum_{r=0}^{n-2}\alpha^r B_{n-r} \qquad (50)$$

$$n = 2, 3, 4, \ldots$$

Where

$$\alpha = \frac{T}{12}\,\frac{5+8z-z^2}{1-z}, \text{ and } B = \frac{T}{12}\left(\frac{1}{4}g_1 z + \frac{1}{12}g_2 z - \frac{1}{6}g_2 z^2\right)$$

Checking for accuracy, equation (50) yields

$$Z\left[1/s^3\right] = \frac{T^2 z}{2(1-z)^3}\left(\frac{195}{36} - \frac{19}{18}z + \frac{5}{3}z^2 - \frac{7}{18}z^3 - \frac{1}{36}z^4\right) \qquad (51)$$

as compared with the exact solution

$$Z\left[1/s^3\right] = \frac{T^2 z}{2(1-z)^3}(1+z) \tag{52}$$

This rules out the use of equation (49) as a method of approximating $Z\left[1/s^n\right]$, for it should at least be exact for n=2 and 3 to be useful.

The Exact Solution of $Z\left[1/s^n\right]$

Criswell (2) presented an algorithm for determining the Z-transform of $t^n/n!$. This algorithm generates tables of Z-transforms of $t^n/n!$. Table 1 of Appendix A can be used to obtain the coefficients $A(n,p)$ in $A_n(z) = \sum_{p=-1}^{n-2} A(n,p)z^p$ for the exact form,

$$Z\,(1/s^n) = \frac{T^{n-1}z}{(n-1)!}\,\frac{A_n(z)}{(1-z)^n}. \quad n=1,\ 2,\ 3,\ 4,\ \ldots\ 15 \tag{53}$$

Z-TRANSFORMS USED IN THE SOLUTION OF DIFFERENTIAL EQUATIONS

Z-Transforms Developed with Trapezoidal Convolution

The following equations were developed by Halijak (1), using equation (22).

$$Z\left[\frac{1}{s}\,\bar{g}\right] \doteq \frac{T}{2}\,\frac{1+z}{1-z}Z\bar{g} - \frac{T}{2}\,\frac{g_0}{1-z} \tag{54}$$

$$Z(1/s^n) \doteq \frac{Tz}{(1-z)^2}\left[\frac{T}{2}\,\frac{1+z}{1-z}\right]^{n-2} \qquad n \geq 2 \tag{55}$$

$$Z\left[\frac{1}{s}\,\bar{g}\right] \doteq \frac{T^2 z}{(1-z)^2}\left[\frac{T}{2}\,\frac{1+z}{1-z}\right]^{n-2}\left(Z\bar{g} - \frac{1}{2}g_0\right) \qquad n \geq 2 \tag{56}$$

Z-Transforms Developed from the Modified Simpson's Convolution

Using Criswell's exact $Z\left[1/s^n\right]$ equation (49) yields

$$Z\left[\frac{1}{s}\,\bar{g}\right] \doteq \frac{T}{1-z}\left[\left(\frac{5}{12}+\frac{2}{3}z-\frac{1}{12}z^2\right)Z\bar{g}-\frac{5}{12}g_0-\left(\frac{1}{4}g_0-\frac{1}{4}g_1+\frac{1}{12}g_2\right)z\right] \quad (57)$$

$$Z\left[\frac{1}{s^2}\,\bar{g}\right] \doteq \frac{T^2}{(1-z)^2}\left[\left(\frac{13}{12}z-\frac{1}{6}z^2+\frac{1}{12}z^3\right)Z\bar{g}-\right.$$

$$\left.\left(\frac{2}{3}g_0-\frac{1}{12}g_2\right)z+\left(\frac{1}{4}g_1-\frac{1}{6}g_2\right)z^2\right] \quad (58)$$

$$Z\left[\frac{1}{s^3}\,\bar{g}\right] \doteq \frac{T^3}{2(1-z)^3}\left[\left(\frac{13}{12}z+\frac{3}{4}z^2+\frac{1}{4}z^3-\frac{1}{12}z^4\right)Z\bar{g}-\left(\frac{2}{3}g_0+\frac{1}{12}g_2\right)z-\right.$$

$$\left.\left(\frac{2}{3}g_0-\frac{1}{4}g_1-\frac{1}{4}g_2\right)z^2+\left(\frac{1}{4}g_1-\frac{1}{3}g_2\right)z^3\right] \quad (59)$$

In general

$$Z\left[\frac{1}{s^n}\,\bar{g}\right] = T\left(Z\bar{g}-\frac{2}{3}g_0+\frac{1}{4}g_1z-\frac{1}{12}g_2z^2\right)\left(\frac{T^{n-1}z}{(n-1)!}\;\frac{A_n(z)}{(1-z)^n}\right)$$

$$+\left(\frac{T}{12}\;\frac{T^{n-1}z}{(n-1)!}\right)Z\bar{g}-\left(\frac{T}{12}\;\frac{(-T)^{n-1}}{(n-1)!}\right)g_2z \quad (60)$$

Z-Transforms Developed from the Averaged Modified Simpson's Convolution

Using the symmetric property of the convolution integral, taking the arithmetic average of the results, the following formulas were developed:

$$Z\left[\overline{fg}\right] = T(Z\bar{f})(Z\bar{g}) - \frac{T}{2}\left(\frac{5}{4}f_0 - \frac{1}{3}f_1 z + \frac{1}{12}f_2 z^2\right)Z\bar{g} - \frac{T}{2}\left(\frac{5}{4}g_0 - \frac{1}{3}g_1 z + \frac{1}{12}g_2 z^2\right)Z\bar{f} +$$

$$\frac{T}{4}f_0 g_0 - \frac{T}{24}(g_{-1}f_2 + f_{-1}g_2)z \qquad (61)$$

$$Z\left[\frac{1}{s}\,\bar{g}\right] = \frac{T}{1-z}\left[\left(\frac{3}{8} + \frac{19}{24}z - \frac{5}{24}z^2 + \frac{1}{24}z^3\right)Z\bar{g} - \frac{3}{8}g_0 - \right.$$

$$\left.\left(\frac{1}{4}g_0 - \frac{1}{6}g_1 + \frac{1}{24}g_{-1} + \frac{1}{24}g_2\right)z + \frac{1}{24}g_{-1}z^2\right] \qquad (62)$$

$$Z\left[\frac{1}{s^2}\,\bar{g}\right] = \frac{T^2}{(1-z)^2}\left[\left(\frac{7}{6}z - \frac{5}{12}z^2 + \frac{1}{3}z^3 - \frac{1}{12}z^4\right)Z\bar{g} - \left(\frac{5}{8}g_0 + \frac{1}{12}g_{-1} - \frac{1}{24}g_2\right)z + \right.$$

$$\left.\left(\frac{1}{6}g_1 + \frac{1}{6}g_{-1} - \frac{1}{12}g_2\right)z^2 - \frac{1}{12}z^3\right] \qquad (63)$$

$$Z\left[\frac{1}{s^3}\,\bar{g}\right] = \frac{T^3}{2(1-z)^3}\left[\left(\frac{7}{6}z + \frac{1}{3}z^2 + z^3 - \frac{2}{3}z^4 + \frac{1}{6}z^5\right)Z\bar{g} - \right.$$

$$\left(\frac{5}{8}g_0 + \frac{1}{6}g_{-1} + \frac{1}{24}g_2\right)z - \left(\frac{5}{8}g_0 - \frac{1}{6}g_1 - \frac{1}{2}g_{-1} - \frac{1}{8}g_2\right)z^2 +$$

$$\left.\left(\frac{1}{6}g_1 - \frac{1}{6}g_2 - \frac{1}{2}g_{-1}\right)z^3 + \frac{1}{6}g_{-1}z^4\right] \qquad (64)$$

Since there were no original equations to check equations (57) through (64) an effort was made to find some simple check. It was observed, after deriving the first two equations, that the numerical coefficients of $Z\bar{g}$ added to 1 in each case and the rest of the numerical coefficients added to -1/2. Using this as a guide it was found that this was also true for all of the equations developed thereafter, thus enabling a simple check for accuracy.

The Averaged Modified Simpson's Convolution formulas will not be used in the numerical analysis since they contain two negative subscripted variables, of which only one can be found. It should also be noted that

these formulas create recursive relations which are one order higher than those obtained by the Modified Simpson's Convolution formulas. This could possibly make the numerical data more susceptible to round-off error and spurious solutions.

NUMERICAL EXAMPLES

The following differential equations are solved using Trapezoidal Convolution and the Modified Simpson's Convolution formulas:

$$\frac{dy}{dt} + y = 1 \qquad y(0) = 0$$

$$\frac{d^2y}{dt^2} + \frac{2dy}{dt} + y = 1 \qquad \dot{y}(0) = y(0) = 0$$

$$\frac{d^3y}{dt^3} + \frac{3d^2y}{dt^2} + \frac{3dy}{dt} + y = 1 \qquad \ddot{y}(0) = \dot{y}(0) = y(0) = 0$$

$$\frac{dy}{dt} + y^2 = 1 \qquad y(0) = 0$$

Only the equations, the type of solution used, the recurrence relations, and pertinent data are shown. The step-by-step solutions and computer programs are shown in the Appendices.

Empirical Error Analysis

A comparison of the computational errors generated by various sampling sizes when using Trapezoidal Convolution, the Modified Simpson's Convolution and Simpson's 1/3 rule is presented for the first order differential equation. The data for Simpson's 1/3 rule was taken from Lei's Master's Report (11).

Recursive Formulas

a) Solution of $\frac{dy}{dt} + y = 1$, $y(0) = 1$ (Appendix B)

Exact solution,

$$y(t) = 1-e^{-t} \tag{65}$$

Trapezoidal Convolution,

$$y_n = \frac{TX_n - (\frac{T}{2} - 1)y_{n-1}}{(1 + \frac{T}{2})} \qquad \left\{X_n\right\} = \left\{0,\ 1,\ 1,\ 1,\ 1,\ \ldots\right\} \tag{66}$$

Modified Simpson's Convolution,

$$y_n = \frac{X_n - (\frac{2}{3}T - 1)y_{n-1} + (\frac{T}{12})y_{n-2}}{(1 + \frac{5}{12}T)} \qquad \left\{X_n\right\} = \left\{0, T(1 + \frac{y_2}{12} - \frac{y_1}{4}), T, T, \ldots\right\} \tag{67}$$

where

$$y_1 = \frac{T(1 + \frac{T}{2})}{1+T+ \frac{T^2}{3}}, \qquad y_2 = \frac{2T}{1+T+ \frac{T^2}{3}}$$

b) Solution of $\frac{d^2y}{dt^2} + \frac{2dy}{dt} + y = 1$, $\dot{y}(0) = y(0) = 0$ (Appendix C)

Exact solution,

$$y(t) = 1-e^{-t}-te^{-t} \tag{68}$$

Trapezoidal Convolution,

$$y_n = \frac{T^2X_n - (T^2-2)y_{n-1} - (1-T)y_{n-2}}{(T+1)} \qquad \left\{X_n\right\} = \left\{0,\ \frac{1}{2},\ 1,\ 1,\ 1,\ \ldots\right\} \tag{69}$$

Modified Simpson's Convolution,

$$y_n = \frac{X_n - (\frac{13T^2}{12} - 2 + \frac{T}{2})y_{n-1} - (1 - \frac{T^2}{6} - \frac{3}{2}T)y_{n-2} - T(\frac{T}{12} + \frac{1}{6})y_{n-3}}{(1 + \frac{5T}{6})} \qquad (70)$$

$$\left\{X_n\right\} = \left\{0, T(\frac{T}{2} - \frac{Ty_2}{12} - \frac{y_1}{2} + \frac{y_2}{6}),\ T(T - \frac{Ty_1}{4} + \frac{Ty_2}{6} + \frac{y_1}{2} - \frac{y_2}{6}),\ T^2, T^2, \ldots\right\}$$

where

$$y_1 = \frac{T^2(\frac{1}{2} + \frac{2T}{3} - \frac{T^2}{6})}{1 + 2T + \frac{4T^2}{3} \quad \frac{T^4}{9}}, \qquad y_2 = \frac{T^2(2 + \frac{4T}{3} \quad \frac{2T^2}{3})}{1 + 2T + \frac{4T^2}{3} \quad \frac{T^4}{9}}$$

c) Solution of $\frac{d^3y}{dt^3} + \frac{3d^2y}{dt^2} + \frac{3dy}{dt} + y = 1,\ \ddot{y}(0) = \dot{y}(0) = \bar{y}(0) = 0$

(Appendix D)

Exact solution,

$$y(t) = 1 - e^{-t} - te^{-t} - t^2/2\ e^{-t} \qquad (71)$$

Trapezoidal Convolution,

$$y_n = \frac{T^3X_n - (\frac{T^2}{3} - 3 + \frac{3T}{2})y_{n-1} - (3 - \frac{9T}{2} + \frac{T^2}{2})y_{n-2} - (\frac{3T}{2} - 1)y_{n-3}}{(1 + \frac{3T}{2})} \qquad (72)$$

$$\left\{X_n\right\} = \left\{0, \frac{1}{4}, \frac{3}{4}, 1, 1, 1, \ldots\right\}$$

Modified Simpson's Convolution,

$$y_n = \Big[X_n - (\frac{13T^2}{4} - \frac{T}{2} - 3 + \frac{13T^3}{24})y_{n-1} - (3 - 3T - \frac{15T^2}{4} + \frac{3T^3}{8})y_{n-2}$$

$$- (\frac{5T}{2} - 1 + \frac{3T^2}{4} + \frac{T^3}{8})y_{n-3} + T(\frac{1}{4} + \frac{T}{4} + \frac{T^2}{24})y_{n-4}\Big] \div (1 + \frac{5T}{4}) \qquad (73)$$

$$\left\{X_n\right\}=\left\{0,T(\frac{y_2}{4}-\frac{Ty_2}{4}+\frac{T^2}{4}-\frac{3y_1}{4}+\frac{T^2y_2}{24}),\ T(\frac{3T^2}{4}+\frac{3Ty_2}{4}-\frac{y_2}{2}+\frac{3y_1}{2}\right.$$

$$-\frac{3Ty_1}{4}-\frac{T^2y}{8}-\frac{T^3y_2}{8}),\ T(T^2-\frac{3y_1}{4}+\frac{3Ty_1}{4}-\frac{T^2y_1}{8}+\frac{y_2}{4}-\frac{Ty_2}{2}+\frac{T^2y_2}{6}),$$

$$\left.T^3,\ T^3,\ T^3,\ \ldots\right\}$$

where

$$y_1=\frac{T^3(\frac{1}{4}+\frac{T}{4}+\frac{T^3}{16})}{1+\frac{9T}{2}+\frac{5T^2}{2}-3T^3+T^4+\frac{T^5}{3}+\frac{T^6}{36}},\quad y_2=\frac{T^3(\frac{3}{2}+\frac{7T}{4}-T^2-\frac{T^3}{3})}{1+\frac{9T}{2}+\frac{5T^2}{2}-3T^3+T^4+\frac{T^5}{3}+\frac{T^6}{36}}$$

d) Solution of $\frac{dy}{dt}+y^2=1,\ y(0)=0$ (Appendix E)

Exact solution,

$$y(t)=1-\frac{2}{1+e^{2t}} \tag{74}$$

Trapezoidal Convolution,

$$y_n=\frac{-1+\sqrt{1-2T(\frac{T}{2}y^2_{n-1}-y_{n-1}-T)}}{T} \qquad n\geq 1 \tag{75}$$

Modified Simpson's Convolution,

$$y_n=\frac{-1+\sqrt{1-\frac{5T}{3}(\frac{2T}{3}y^2_{n-1}-y_{n-1}-\frac{T}{12}y^2_{n-2}-T)}}{\frac{5T}{6}} \qquad n\geq 1 \tag{76}$$

where

$$y_1=T-\frac{T^3}{3}+\frac{2T^5}{15}-\frac{5.67T^7}{105}$$

$$y_2=2T-\frac{8}{3}T^3+\frac{64T^5}{15}-\frac{726T^7}{105}$$

Results

The error measurement for the first order equation is presented in Table 2, and graphed in Figs. 2 through 6. Decimal place accuracy for various sampling intervals for the second order, third order, and non-linear differential equations is presented in Tables 3 through 14.

Table 2. E_{max} for $\frac{dy}{dt} + y = 1$, $y(0) = 0$, solution time (t) = .8 sec.

T	Trap. Conv.	Mod. Simp. Conv.	Simpson's 1/3 Rule
.001	.00000015	.00002292	-
.002	.00000025	.00000496	-
.003	.00000110	.00000092	.00000038
.004	.00000055	.00000567	.00000039
.005	.00000070	.00000202	.00000014
.006	.00000087	.00000050	.00000027
.007	.00000097	.00000325	.00000031
.008	.00000142	.00000122	.00000015
.009	.00000235	.00000031	.00000014
.010	.00000253	.00000224	.00000012
.012	.00000389	.00000029	.00000014
.014	.00000582	.00000064	.00000008
.016	.00000738	.00000129	.00000018
.018	.00000960	.00000031	.00000015
.020	.00001191	.00000036	.00000015
.022	.00001442	.00000084	.00000015
.024	.00001708	.00000034	.00000013
.026	.00001996	.00000013	.00000014
.028	.00002332	.00000046	.00000015
.030	.00002654	.00000051	.00000014
.032	.00002269	.00000040	.00000011
.034	.00003443	.00000023	.00000011
.036	.00003863	.00000074	.00000011
.038	.00004320	.00000058	.00000011
.040	.00004779	.00000049	.00000010
.042	.00005278	.00000110	.00000013
.044	.00005778	.00000094	.00000009
.046	.00006300	.00000092	.00000013
.048	.00006841	.00000154	.00000016
.050	.00007475	.00000154	.00000006
.052	.00008059	.00000150	.00000012
.054	.00008624	.00000213	.00000006
.056	.00009346	.00000216	.00000008
.058	.00009941	.00000220	.00000008
.060	.00010724	.00000285	.00000014
.062	.00011324	.00000294	.00000010
.064	.00012163	.00000302	.00000008
.066	.00013022	.00000374	.00000008
.068	.00013646	.00000383	.00000010
.070	.00014559	.00000405	.00000007
.072	.00015499	.00000478	.00000009
.074	.00016121	.00000483	.00000012
.076	.00017113	.00000514	.00000012
.078	.00018138	.00000592	.00000015

Table 2. (cont.)

T	Trap. Conv.	Mod. Simp. Conv.	Simpson's 1/3 Rule
.080	.00019174	.00000632	.00000006
.082	.00019785	.00000634	.00000012
.084	.00020885	.00000719	.00000009
.086	.00022012	.00000759	.00000010
.088	.00023164	.00000804	.00000015
.090	.00023673	.00000847	.00000013
.092	.00024894	.00000890	.00000011
.094	.00026122	.00000951	.00000010
.096	.00027391	.00001038	.00000015
.098	.00028673	.00001104	.00000022
0.100	.00029981	.00001169	.00000016

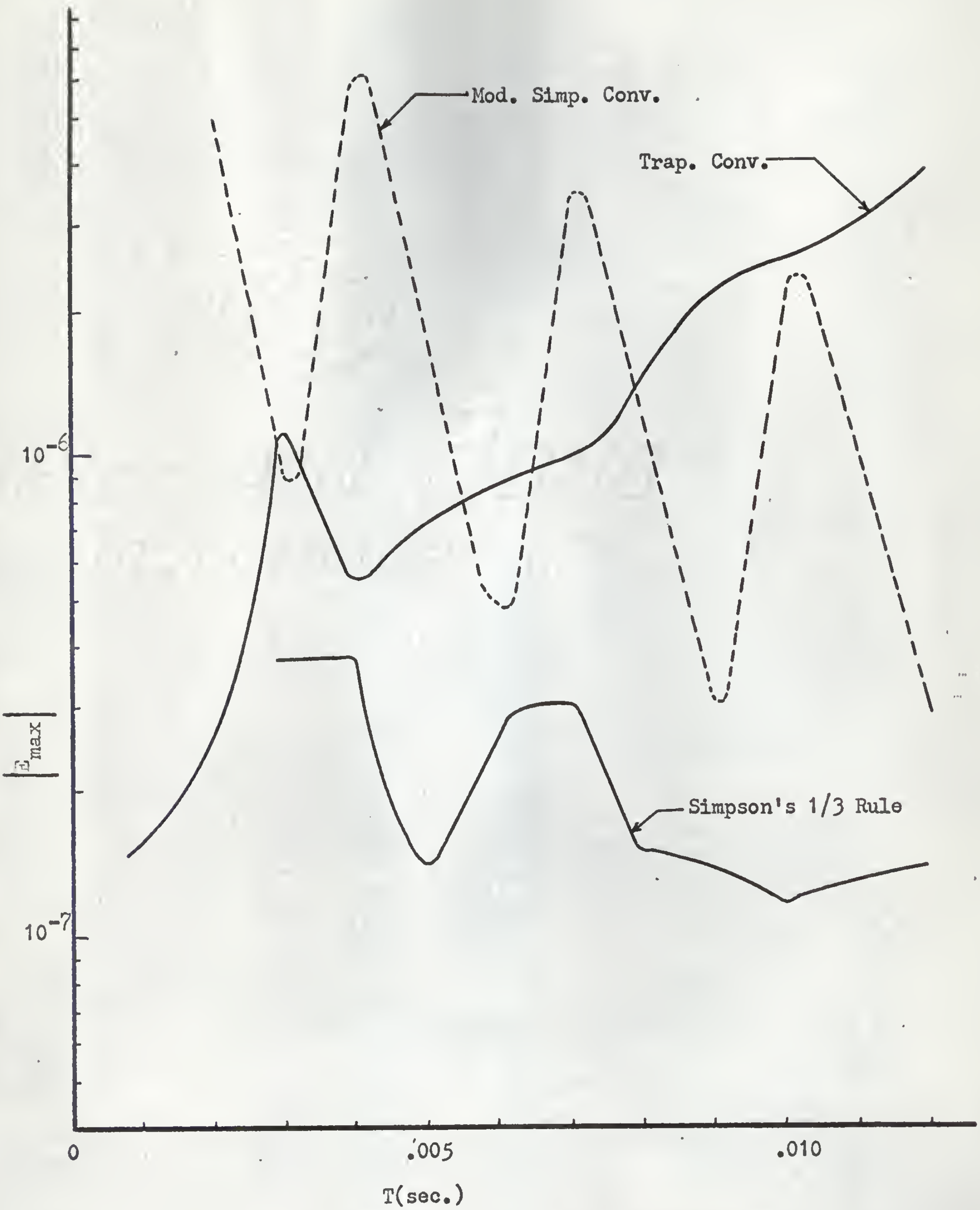

Fig. 2. Error curves of $\frac{dy}{dt} + y = 1$, $y(0) = 0$, solution time, (t) = .8 sec.

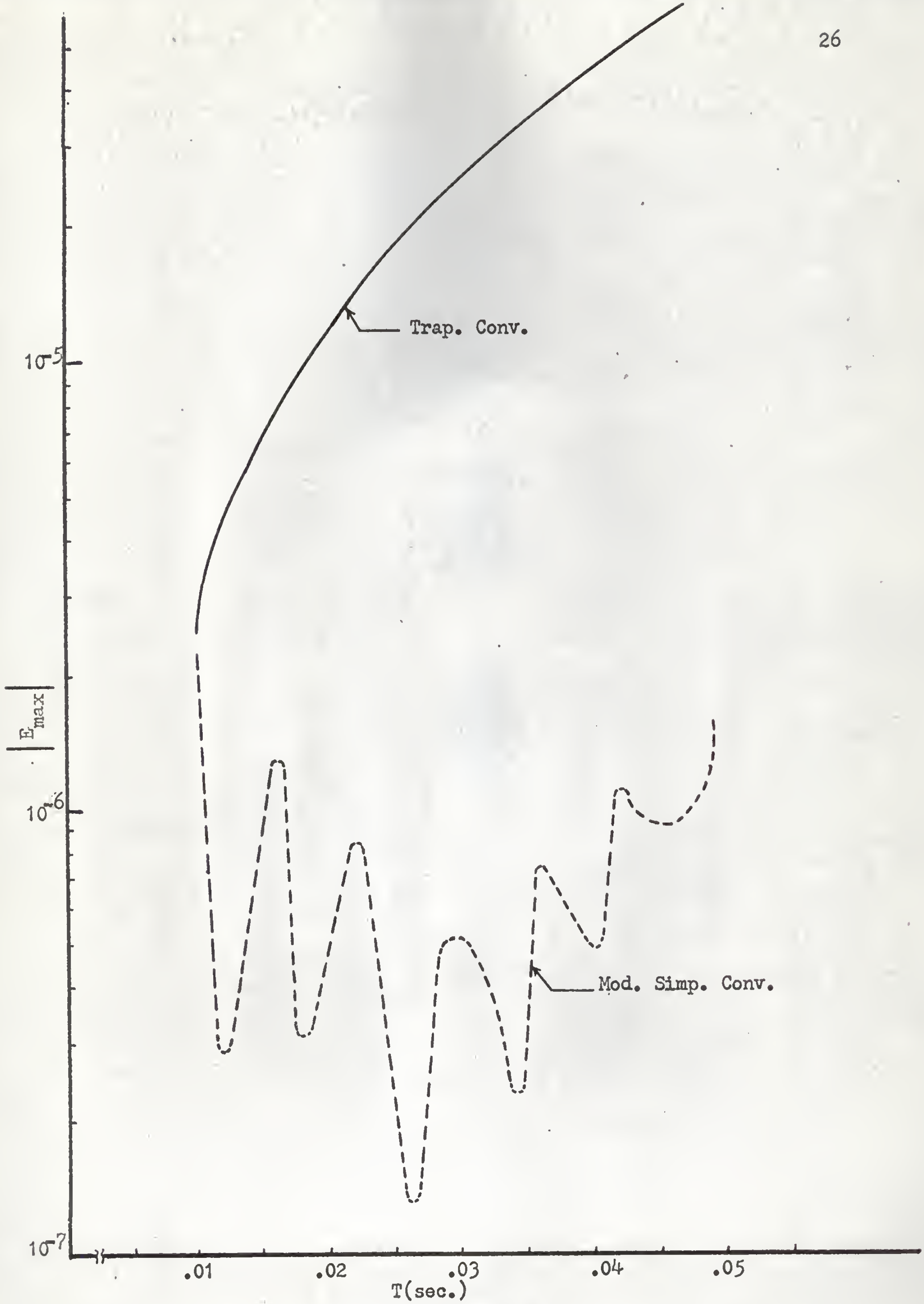

Fig. 3. Error curves of $\frac{dy}{dt} + y = 1$, $y(0) = 0$, solution time, (t) = .8 sec.

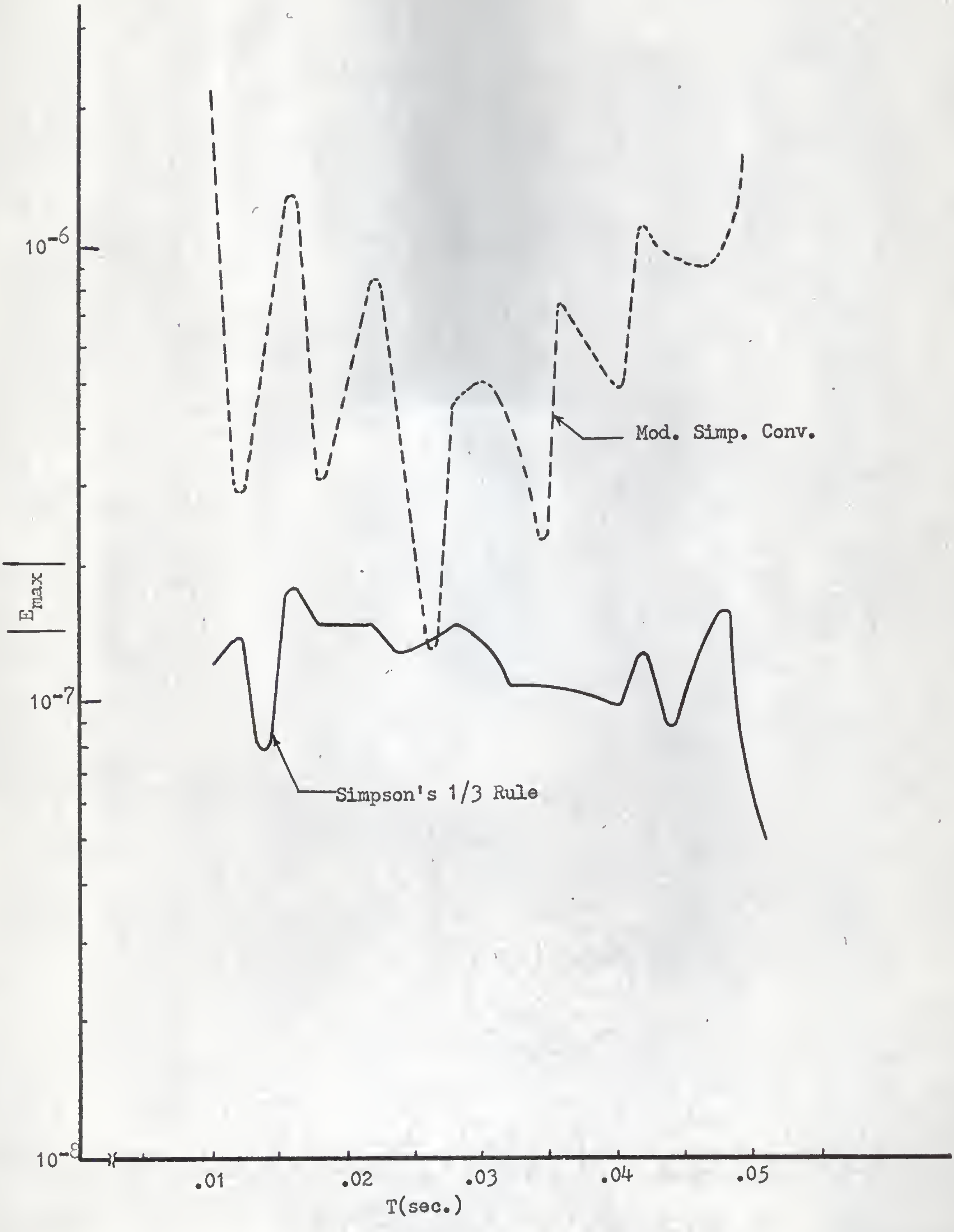

Fig. 4. Error curves of $\frac{dy}{dt} + y = 1$, $y(0) = 0$, solution time, $(t) = .8$ sec.

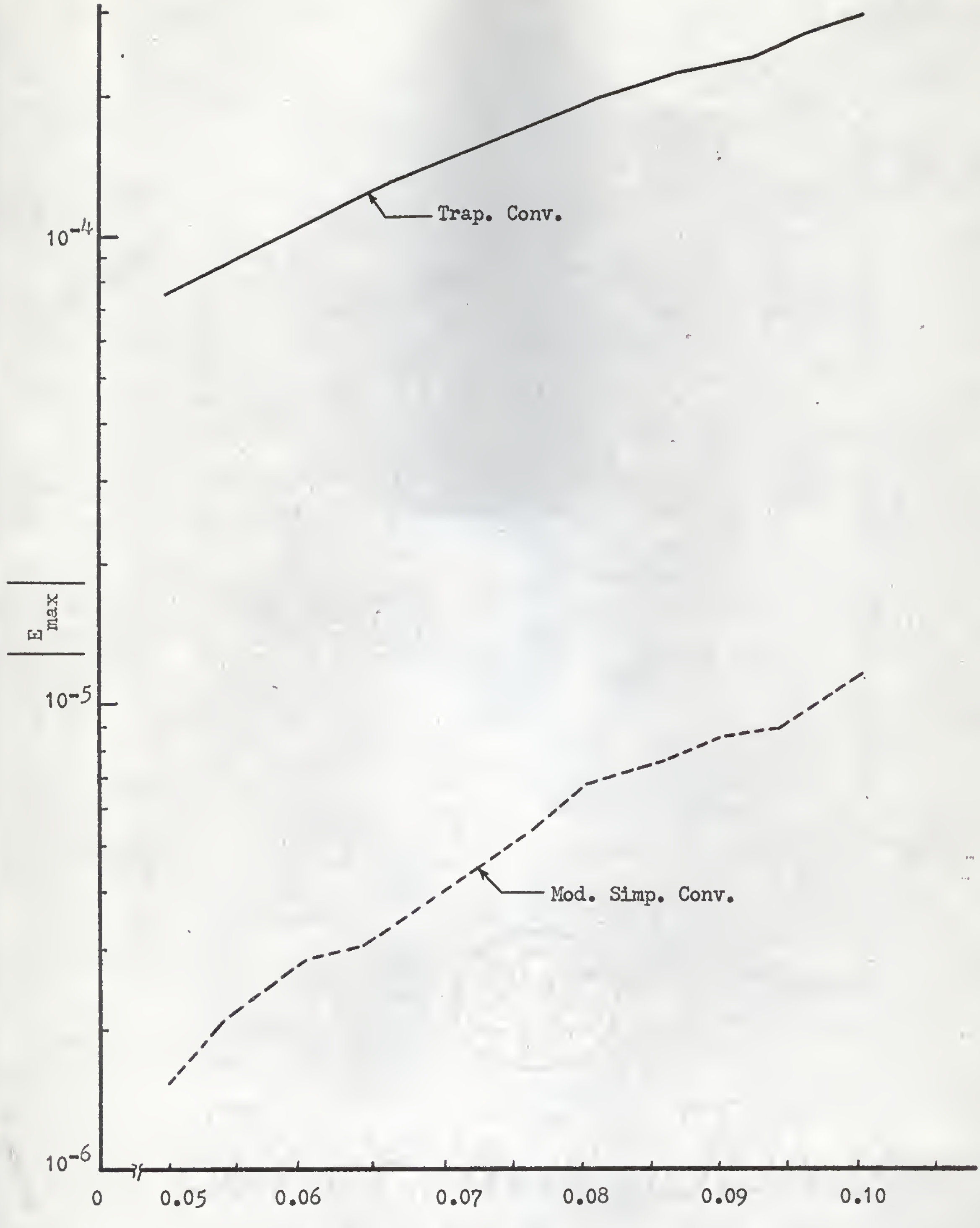

Fig. 5. Error curves of $\frac{dy}{dt} + y = 1$, $y(0) = 0$, solution time, $(t) = .8$ sec.

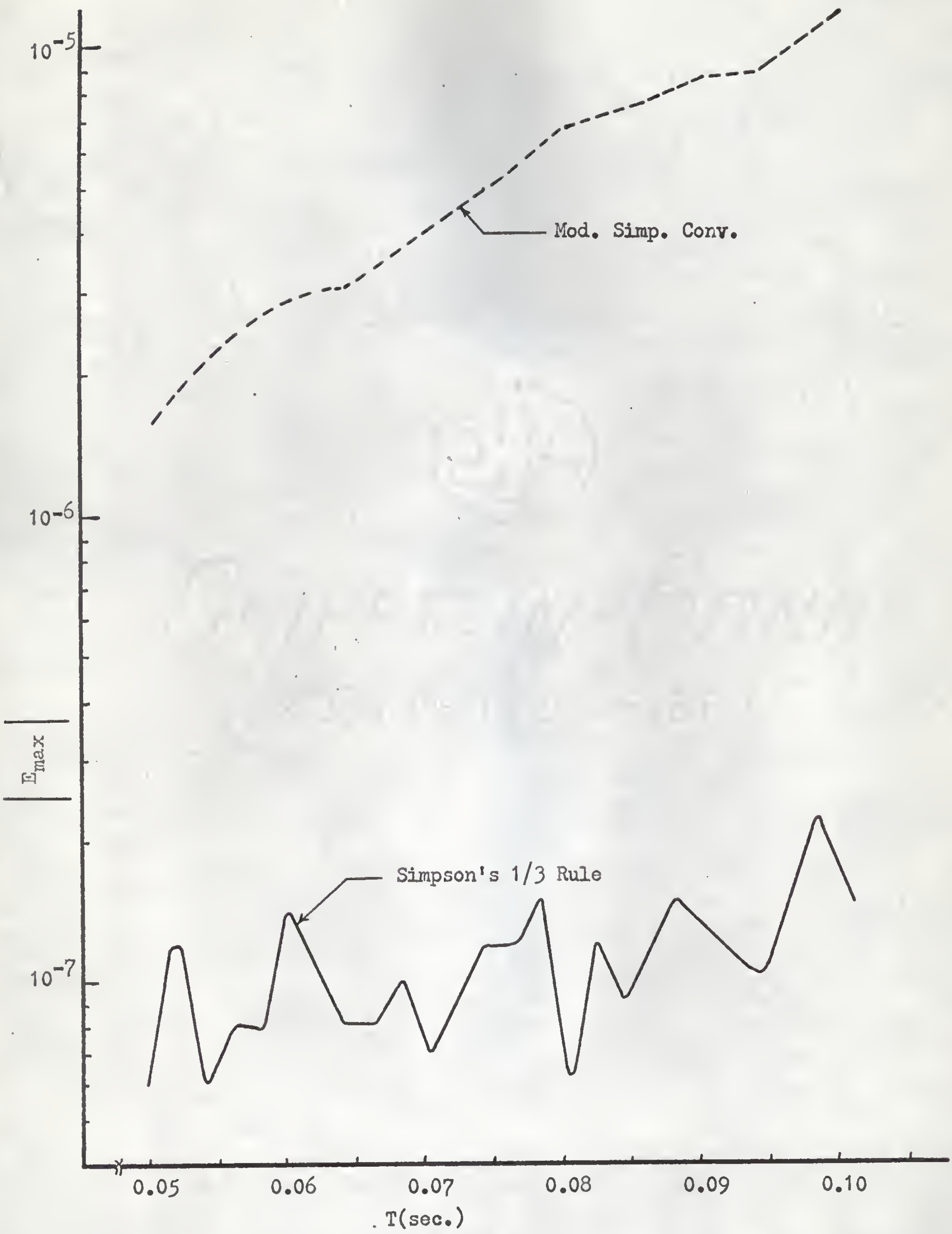

Fig. 6. Error curves of $\frac{dy}{dt} + y = 1$, $y(0) = 0$, solution time, $(t) = .8$ sec.

Table 3.

SOLUTION OF D(2)Y/DT(2)+2DY/DT+Y=1,MODIFIED SIMPSONS CONV.

TIME	EXACT	APPROX.	ERROR	REL. ERROR
.003	.00000439	.00000449	.00000010	2.38955360
.004	.00000787	.00000798	.00000011	1.41212190
.005	.00001244	.00001246	.00000002	.16816080
.006	.00001789	.00001793	.00000004	.20215180
.007	.00002433	.00002439	.00000006	.23626200
.008	.00003174	.00003183	.00000009	.27049206
.009	.00004024	.00004026	.00000002	.05330745
.010	.00004960	.00004967	.00000007	.13499325
.011	.00006004	.00006006	.00000002	.03535487
.012	.00007134	.00007143	.00000009	.12430682
.013	.00008371	.00008377	.00000007	.07926767
.014	.00009703	.00009709	.00000006	.05800132
.015	.00011132	.00011138	.00000006	.05526410
.016	.00012656	.00012664	.00000008	.06409507
.017	.00014286	.00014287	.00000002	.01127701
.018	.00016000	.00016007	.00000007	.04346250
.019	.00017819	.00017823	.00000004	.02215039
.020	.00019733	.00019735	.00000003	.01416438
.021	.00021740	.00021744	.00000004	.01703764
.022	.00023841	.00023848	.00000007	.02843805
.023	.00026046	.00026048	.00000002	.00702598
.024	.00028344	.00028343	.00000000	.00260724
.025	.00030735	.00030734	.00000001	.00331867
.026	.00033219	.00033220	.00000001	.00419644
.027	.00035805	.00035801	.00000004	.01114379
.028	.00038472	.00038476	.00000004	.00997341
.029	.00041242	.00041246	.00000004	.00990010
.030	.00044113	.00044110	.00000003	.00667146
.031	.00047065	.00047069	.00000003	.00715600
.032	.00050129	.00050121	.00000008	.01551002
.033	.00053263	.00053267	.00000005	.00870592
.034	.00056507	.00056507	.00000000	.00004424
.035	.00059831	.00059840	.00000009	.01552380
.036	.00063265	.00063266	.00000001	.00228879
.037	.00066788	.00066786	.00000003	.00403365
.038	.00070401	.00070398	.00000003	.00444598
.039	.00074102	.00074102	.00000000	.00047502
.040	.00077892	.00077899	.00000007	.00946182
.041	.00081791	.00081789	.00000002	.00216773
.042	.00085767	.00085770	.00000004	.00412981
.043	.00089841	.00089843	.00000003	.00314335
.044	.00094002	.00094008	.00000006	.00637750
.045	.00098261	.00098265	.00000004	.00382043
.046	.00102607	.00102613	.00000006	.00596453
.047	.00107049	.00107052	.00000002	.00215789
.048	.00111578	.00111581	.00000004	.00335193

Table 3. (cont.)

TIME	EXACT	APPROX.	ERROR	REL. ERROR
.049	.00116202	.00116202	.00000000	.00024096
.050	.00120902	.00120913	.00000011	.00880876
.051	.00125708	.00125715	.00000006	.00489227
.052	.00130600	.00130606	.00000007	.00501533
.053	.00135586	.00135588	.00000002	.00165947
.054	.00140657	.00140659	.00000003	.00204043
.055	.00145822	.00145821	.00000000	.00065834
.056	.00151060	.00151071	.00000011	.00718918
.057	.00156414	.00156411	.00000003	.00163029
.058	.00161830	.00161840	.00000010	.00638325
.059	.00167350	.00167358	.00000008	.00490589
.060	.00172963	.00172965	.00000002	.00116210
.061	.00178658	.00178660	.00000002	.00086198
.062	.00184436	.00184443	.00000007	.00402308
.063	.00190306	.00190315	.00000009	.00471346
.064	.00196267	.00196275	.00000007	.00361751
.065	.00202321	.00202322	.00000000	.00027184
.066	.00208456	.00208457	.00000000	.00039817
.067	.00214672	.00214680	.00000008	.00366140
.068	.00220989	.00220989	.00000000	.00030318
.069	.00227375	.00227386	.00000011	.00486421
.070	.00233863	.00233870	.00000007	.00310866
.071	.00240430	.00240441	.00000010	.00424655
.072	.00247087	.00247098	.00000010	.00420904
.073	.00253834	.00253841	.00000007	.00293499
.074	.00260659	.00260671	.00000011	.00436969
.075	.00267574	.00267586	.00000013	.00483605
.076	.00274576	.00274588	.00000012	.00423562
.077	.00281658	.00281675	.00000017	.00620612
.078	.00288837	.00288848	.00000011	.00366989
.079	.00296095	.00296106	.00000011	.00375894
.080	.00303429	.00303449	.00000020	.00655508
.081	.00310861	.00310877	.00000016	.00521134
.082	.00318369	.00318389	.00000020	.00636996
.083	.00325976	.00325987	.00000011	.00348492
.084	.00333647	.00333669	.00000022	.00658181
.085	.00341415	.00341435	.00000020	.00573788
.086	.00349269	.00349285	.00000015	.00440634
.087	.00357199	.00357219	.00000020	.00562432
.088	.00365214	.00365237	.00000022	.00613339
.089	.00373314	.00373338	.00000024	.00636194
.090	.00381499	.00381522	.00000024	.00620448
.091	.00389758	.00389790	.00000032	.00825641
.092	.00398122	.00398141	.00000019	.00467444
.093	.00406550	.00406575	.00000024	.00602140
.094	.00415062	.00415091	.00000029	.00706401
.095	.00423667	.00423690	.00000023	.00546420
.096	.00432345	.00432371	.00000026	.00603916
.097	.00441106	.00441134	.00000028	.00639755
.098	.00449950	.00449980	.00000030	.00655628
.099	.00458876	.00458907	.00000030	.00660091

ERROR LC-2 IN STATEMENT 0001 + 00 LINES

Table 4.

SOLUTION OF D(2)Y/DT(2)+2DY/DT+Y=1,MODIFIED SIMPSONS CONV.

TIME	EXACT	APPROX.	ERROR	REL. ERROR
.030	.00044113	.00044110	.00000003	.00672813
.040	.00077892	.00077899	.00000007	.00884686
.050	.00120902	.00120911	.00000009	.00733650
.060	.00172963	.00172961	.00000002	.00117366
.070	.00233863	.00233863	.00000000	.00004704
.080	.00303429	.00303436	.00000008	.00247175
.090	.00381499	.00381503	.00000004	.00107733
.100	.00467886	.00467886	.00000000	.00013037
.110	.00562414	.00562414	.00000000	.00007468
.120	.00664914	.00664914	.00000000	.00000902
.130	.00775208	.00775220	.00000012	.00150411
.140	.00893154	.00893165	.00000011	.00122599
.150	.01018580	.01018587	.00000007	.00067741
.160	.01151319	.01151325	.00000006	.00054720
.170	.01291218	.01291222	.00000004	.00030979
.180	.01438115	.01438121	.00000006	.00044503
.190	.01591855	.01591870	.00000015	.00096114
.200	.01752314	.01752318	.00000004	.00021686
.210	.01919310	.01919315	.00000005	.00028135
.220	.02092706	.02092717	.00000011	.00052564
.230	.02272367	.02272379	.00000012	.00050608
.240	.02458151	.02458158	.00000007	.00028883
.250	.02649910	.02649916	.00000006	.00023774
.260	.02847497	.02847516	.00000019	.00065672
.270	.03050803	.03050821	.00000018	.00059001
.280	.03259673	.03259699	.00000026	.00080069
.290	.03473995	.03474019	.00000024	.00069660
.300	.03693632	.03693652	.00000020	.00054689
.310	.03918442	.03918471	.00000029	.00074519
.320	.04148330	.04148351	.00000021	.00051105
.330	.04383135	.04383169	.00000034	.00078026
.340	.04622777	.04622804	.00000027	.00058839
.350	.04867106	.04867137	.00000031	.00063898
.360	.05116020	.05116051	.00000031	.00059812
.370	.05369398	.05369430	.00000032	.00059225
.380	.05627125	.05627162	.00000037	.00065398
.390	.05889090	.05889134	.00000044	.00074375
.400	.06155196	.06155236	.00000040	.00064823
.410	.06425318	.06425360	.00000042	.00065522
.420	.06699353	.06699401	.00000048	.00071201
.430	.06977197	.06977253	.00000056	.00079545
.440	.07258757	.07258813	.00000056	.00076735
.450	.07543921	.07543980	.00000059	.00078606
.460	.07832591	.07832655	.00000064	.00082093
.470	.08124662	.08124740	.00000078	.00095512
.480	.08420055	.08420137	.00000082	.00097862
.490	.08718664	.08718753	.00000089	.00102539
.500	.09020405	.09020495	.00000090	.00099330

Table 4. (cont.)

TIME	EXACT	APPROX.	ERROR	REL. ERROR
.510	.09325164	.09325269	.00000105	.00112599
.520	.09632880	.09632987	.00000107	.00110559
.530	.09943434	.09943558	.00000124	.00125007
.540	.10256773	.10256898	.00000125	.00121871
.550	.10572780	.10572920	.00000140	.00132416
.560	.10891380	.10891539	.00000159	.00145987
.570	.11212509	.11212671	.00000162	.00144481
.580	.11536052	.11536235	.00000183	.00158633
.590	.11861970	.11862150	.00000180	.00151745
.600	.12190140	.12190338	.00000198	.00162426
.610	.12520505	.12520721	.00000216	.00172517
.620	.12853004	.12853222	.00000218	.00169610
.630	.13187535	.13187768	.00000233	.00176682
.640	.13524043	.13524285	.00000242	.00178941
.650	.13862454	.13862700	.00000246	.00177458
.660	.14202682	.14202943	.00000261	.00183768
.670	.14544675	.14544944	.00000269	.00184947
.680	.14888344	.14888634	.00000290	.00194783
.690	.15233650	.15233946	.00000296	.00194307
.700	.15580497	.15580813	.00000316	.00202818
.710	.15928841	.15929170	.00000329	.00206544
.720	.16278616	.16278954	.00000338	.00207634
.730	.16629754	.16630101	.00000347	.00208662
.740	.16982178	.16982549	.00000371	.00218464
.750	.17335858	.17336238	.00000380	.00219199
.760	.17690710	.17691108	.00000398	.00224977
.770	.18046693	.18047101	.00000408	.00226080
.780	.18403730	.18404159	.00000429	.00233105
.790	.18761781	.18762226	.00000445	.00237184
.800	.19120791	.19121247	.00000456	.00238484
.810	.19480696	.19481168	.00000472	.00242291
.820	.19841444	.19841934	.00000490	.00246958
.830	.20202989	.20203492	.00000503	.00248973
.840	.20565265	.20565792	.00000527	.00256257
.850	.20928238	.20928783	.00000545	.00260414
.860	.21291849	.21292415	.00000566	.00265829
.870	.21656063	.21656640	.00000577	.00266438
.880	.22020812	.22021410	.00000598	.00271561
.890	.22386066	.22386678	.00000612	.00273384
.900	.22751769	.22752398	.00000629	.00276462
.910	.23117873	.23118525	.00000652	.00282033
.920	.23484346	.23485015	.00000669	.00284871
.930	.23851134	.23851824	.00000690	.00289294
.940	.24218202	.24218910	.00000708	.00292342
.950	.24585500	.24586231	.00000731	.00297330
.960	.24952992	.24953746	.00000754	.00302168
.970	.25320644	.25321415	.00000771	.00304495
.980	.25688402	.25689198	.00000796	.00309867
.990	.26056237	.26057057	.00000820	.00314704

ERROR LC-2 IN STATEMENT 0001 + 00 LINES

Table 5.

SOLUTION OF D(2)Y/DT(2)+2DY/DT+Y=1,TRAPEZOIDAL CONVOLUTION

TIME	EXACT	APPROX.	ERROR	REL. ERROR
.0200	.00019733	.00019703	.00000029	.14767441
.0300	.00044113	.00044063	.00000050	.11328330
.0400	.00077892	.00077837	.00000055	.07003415
.0500	.00120902	.00120836	.00000066	.05487071
.0600	.00172963	.00172872	.00000090	.05223673
.0700	.00233863	.00233762	.00000101	.04314923
.0800	.00303429	.00303324	.00000105	.03466052
.0900	.00381499	.00381379	.00000120	.03141033
.1000	.00467886	.00467752	.00000134	.02864161
.1100	.00562414	.00562269	.00000145	.02581195
.1200	.00664914	.00664760	.00000154	.02320601
.1300	.00775208	.00775056	.00000152	.01957668
.1400	.00893154	.00892993	.00000161	.01802937
.1500	.01018580	.01018407	.00000173	.01699425
.1600	.01151319	.01151138	.00000181	.01575584
.1700	.01291218	.01291027	.00000191	.01479224
.1800	.01438115	.01437920	.00000196	.01359418
.1900	.01591855	.01591662	.00000193	.01213050
.2000	.01752314	.01752103	.00000211	.01202410
.2100	.01919310	.01919095	.00000215	.01119673
.2200	.02092706	.02092491	.00000215	.01026900
.2300	.02272367	.02272147	.00000220	.00966393
.2400	.02458151	.02457922	.00000229	.00930781
.2500	.02649910	.02649676	.00000234	.00883426
.2600	.02847497	.02847271	.00000226	.00793328
.2700	.03050803	.03050573	.00000230	.00754883
.2800	.03259673	.03259447	.00000226	.00692094
.2900	.03473995	.03473764	.00000231	.00664077
.3000	.03693632	.03693394	.00000238	.00643269
.3100	.03918442	.03918211	.00000231	.00590286
.3200	.04148330	.04148088	.00000242	.00582885
.3300	.04383135	.04382904	.00000231	.00527020
.3400	.04622777	.04622537	.00000240	.00519168
.3500	.04867106	.04866868	.00000238	.00488997
.3600	.05116020	.05115780	.00000240	.00469310
.3700	.05369398	.05369157	.00000241	.00448095
.3800	.05627125	.05626887	.00000238	.00422774
.3900	.05889090	.05888858	.00000232	.00393270
.4000	.06155196	.06154960	.00000236	.00382766
.4100	.06425318	.06425085	.00000233	.00362472
.4200	.06699353	.06699127	.00000226	.00337794
.4300	.06977197	.06976979	.00000218	.00312160
.4400	.07258757	.07258541	.00000216	.00298123
.4500	.07543921	.07543709	.00000212	.00281154
.4600	.07832591	.07832384	.00000207	.00264025
.4700	.08124662	.08124469	.00000193	.00237179
.4800	.08420055	.08419867	.00000188	.00222920
.4900	.08718664	.08718484	.00000180	.00206224

Table 5. (cont.)

TIME	EXACT	APPROX.	ERROR	REL. ERROR
.5000	.09020405	.09020227	.00000178	.00197663
.5100	.09325164	.09325002	.00000162	.00173723
.5200	.09632880	.09632720	.00000160	.00166305
.5300	.09943434	.09943291	.00000143	.00143713
.5400	.10256773	.10256630	.00000143	.00139420
.5500	.10572780	.10572650	.00000130	.00122957
.5600	.10891380	.10891266	.00000114	.00104670
.5700	.11212509	.11212395	.00000114	.00101672
.5800	.11536052	.11535956	.00000096	.00083217
.5900	.11861970	.11861868	.00000102	.00085989
.6000	.12190140	.12190053	.00000087	.00071369
.6100	.12520505	.12520434	.00000071	.00056707
.6200	.12853004	.12852935	.00000069	.00053684
.6300	.13187535	.13187480	.00000055	.00041706
.6400	.13524043	.13523995	.00000048	.00035492
.6500	.13862454	.13862409	.00000045	.00032462
.6600	.14202682	.14202650	.00000032	.00022531
.6700	.14544675	.14544649	.00000026	.00017876
.6800	.14888344	.14888337	.00000007	.00004702
.6900	.15233650	.15233646	.00000004	.00002626
.7000	.15580497	.15580510	.00000013	.00008344
.7100	.15928841	.15928863	.00000022	.00013811
.7200	.16278616	.16278642	.00000026	.00015972
.7300	.16629754	.16629784	.00000030	.00018040
.7400	.16982178	.16982227	.00000049	.00028854
.7500	.17335858	.17335911	.00000053	.00030572
.7600	.17690710	.17690775	.00000065	.00036742
.7700	.18046693	.18046761	.00000068	.00037680
.7800	.18403730	.18403812	.00000082	.00044556
.7900	.18761781	.18761872	.00000091	.00048503
.8000	.19120791	.19120885	.00000094	.00049161
.8100	.19480696	.19480797	.00000101	.00051846
.8200	.19841444	.19841554	.00000110	.00055440
.8300	.20202989	.20203104	.00000115	.00056922
.8400	.20565265	.20565396	.00000131	.00063700
.8500	.20928238	.20928378	.00000140	.00066895
.8600	.21291849	.21292001	.00000152	.00071389
.8700	.21656063	.21656217	.00000154	.00071112
.8800	.22020812	.22020977	.00000165	.00074929
.8900	.22386066	.22386235	.00000169	.00075493
.9000	.22751769	.22751945	.00000176	.00077357
.9100	.23117873	.23118061	.00000188	.00081322
.9200	.23484346	.23484539	.00000193	.00082182
.9300	.23851134	.23851337	.00000203	.00085111
.9400	.24218202	.24218411	.00000209	.00086299
.9500	.24585500	.24585719	.00000219	.00089077
.9600	.24952992	.24953220	.00000228	.00091372
.9700	.25320644	.25320874	.00000230	.00090835
.9800	.25688402	.25688642	.00000240	.00093427
.9900	.26056237	.26056485	.00000248	.00095179

ERROR LC-2 IN STATEMENT 0001 + 00 LINES

Table 6.

SOLUTION OF D(2)Y/DT(2)+2DY/DT+Y=1,TRAPEZOIDAL CONVOLUTION

TIME	EXACT	APPROX.	ERROR	REL. ERROR
.0020	.00000190	.00000200	.00000010	5.33830050
.0030	.00000439	.00000449	.00000010	2.37822030
.0040	.00000787	.00000798	.00000011	1.40370630
.0050	.00001244	.00001246	.00000002	.16150352
.0060	.00001789	.00001793	.00000004	.19659624
.0070	.00002433	.00002438	.00000006	.23148982
.0080	.00003174	.00003183	.00000008	.26630229
.0090	.00004024	.00004026	.00000002	.04957949
.0100	.00004960	.00004967	.00000007	.13162235
.0110	.00006004	.00006006	.00000002	.03227010
.0120	.00007134	.00007142	.00000009	.12145701
.0130	.00008371	.00008377	.00000006	.07662744
.0140	.00009703	.00009709	.00000005	.05553930
.0150	.00011132	.00011138	.00000006	.05294646
.0160	.00012656	.00012664	.00000008	.06189062
.0170	.00014286	.00014287	.00000001	.00917001
.0180	.00016000	.00016007	.00000007	.04143750
.0190	.00017819	.00017823	.00000004	.02019182
.0200	.00019733	.00019735	.00000002	.01226397
.0210	.00021740	.00021743	.00000003	.01518852
.0220	.00023841	.00023848	.00000006	.02663026
.0230	.00026046	.00026048	.00000001	.00524837
.0240	.00028344	.00028343	.00000001	.00436068
.0250	.00030735	.00030734	.00000002	.00505284
.0260	.00033219	.00033219	.00000000	.00247753
.0270	.00035805	.00035800	.00000005	.01284748
.0280	.00038472	.00038475	.00000003	.00827868
.0290	.00041242	.00041245	.00000003	.00821250
.0300	.00044113	.00044110	.00000004	.00835802
.0310	.00047065	.00047068	.00000003	.00546474
.0320	.00050129	.00050120	.00000009	.01720764
.0330	.00053263	.00053266	.00000004	.00701055
.0340	.00056507	.00056506	.00000000	.00167413
.0350	.00059831	.00059839	.00000008	.01379058
.0360	.00063265	.00063265	.00000000	.00052952
.0370	.00066788	.00066784	.00000004	.00581540
.0380	.00070401	.00070396	.00000004	.00624426
.0390	.00074102	.00074101	.00000000	.00133735
.0400	.00077892	.00077898	.00000006	.00762851
.0410	.00081791	.00081787	.00000003	.00400658
.0420	.00085767	.00085769	.00000002	.00227478
.0430	.00089841	.00089842	.00000001	.00127782
.0440	.00094002	.00094007	.00000004	.00449244
.0450	.00098261	.00098263	.00000002	.00192039
.0460	.00102607	.00102611	.00000004	.00405432
.0470	.00107049	.00107049	.00000000	.00023354
.0480	.00111578	.00111579	.00000002	.00139813

Table 6. (cont.)

TIME	EXACT	APPROX.	ERROR	REL. ERROR
.0490	.00116202	.00116200	.00000003	.00222887
.0500	.00120902	.00120911	.00000008	.00679060
.0510	.00125708	.00125712	.00000004	.00284786
.0520	.00130600	.00130604	.00000004	.00293263
.0530	.00135586	.00135585	.00000000	.00046465
.0540	.00140657	.00140656	.00000000	.00013508
.0550	.00145822	.00145817	.00000004	.00289395
.0560	.00151060	.00151068	.00000007	.00489208
.0570	.00156414	.00156407	.00000006	.00398942
.0580	.00161830	.00161836	.00000006	.00395477
.0590	.00167350	.00167354	.00000004	.00241411
.0600	.00172963	.00172960	.00000002	.00139336
.0610	.00178658	.00178655	.00000003	.00175755
.0620	.00184436	.00184438	.00000002	.00133922
.0630	.00190306	.00190310	.00000004	.00197051
.0640	.00196267	.00196269	.00000002	.00082031
.0650	.00202321	.00202316	.00000005	.00258994
.0660	.00208456	.00208451	.00000005	.00252811
.0670	.00214672	.00214673	.00000001	.00067079
.0680	.00220989	.00220983	.00000006	.00275580
.0690	.00227375	.00227379	.00000004	.00173722
.0700	.00233863	.00233863	.00000000	.00009407
.0710	.00240430	.00240433	.00000002	.00096910
.0720	.00247087	.00247089	.00000002	.00085395
.0730	.00253834	.00253832	.00000001	.00049639
.0740	.00260659	.00260662	.00000002	.00086320
.0750	.00267574	.00267577	.00000003	.00125199
.0760	.00274576	.00274578	.00000002	.00056815
.0770	.00281658	.00281665	.00000007	.00245688
.0780	.00288837	.00288837	.00000000	.00015926
.0790	.00296095	.00296094	.00000000	.00015536
.0800	.00303429	.00303437	.00000008	.00255085
.0810	.00310861	.00310864	.00000003	.00111626
.0820	.00318369	.00318376	.00000007	.00218300
.0830	.00325976	.00325973	.00000003	.00079761
.0840	.00333647	.00333654	.00000007	.00220293
.0850	.00341415	.00341419	.00000004	.00126532
.0860	.00349269	.00349269	.00000000	.00016320
.0870	.00357199	.00357202	.00000003	.00095465
.0880	.00365214	.00365219	.00000005	.00136358
.0890	.00373314	.00373320	.00000006	.00148936
.0900	.00381499	.00381503	.00000005	.00122936
.0910	.00389758	.00389770	.00000012	.00317890
.0920	.00398122	.00398120	.00000002	.00050738
.0930	.00406550	.00406553	.00000003	.00073546
.0940	.00415062	.00415069	.00000007	.00167204
.0950	.00423667	.00423667	.00000000	.00003541
.0960	.00432345	.00432347	.00000002	.00043021
.0970	.00441106	.00441109	.00000003	.00068011
.0980	.00449950	.00449953	.00000003	.00073341
.0990	.00458876	.00458879	.00000003	.00067120

ERROR LC-2 IN STATEMENT 0001 + 00 LINES

Table 7.

SOLUTION OF D(3)Y/DT(3)+3D(2)Y/DT(2)+3DY/DT+Y=1,MOD. SIMP. CONV.

TIME	EXACT	APPROX.	ERROR	REL. ERROR
.0400	.00001029	.00001068	.00000039	3.79652210
.0500	.00001999	.00002049	.00000050	2.50555110
.0600	.00003445	.00003494	.00000049	1.41191240
.0700	.00005426	.00005487	.00000060	1.11429520
.0800	.00008032	.00008110	.00000078	.97599218
.0900	.00011357	.00011442	.00000086	.75470170
.1000	.00015467	.00015560	.00000093	.60307183
.1100	.00020434	.00020538	.00000104	.50750894
.1200	.00026331	.00026447	.00000116	.44023356
.1300	.00033217	.00033357	.00000140	.42027799
.1400	.00041183	.00041334	.00000151	.36725716
.1500	.00050284	.00050443	.00000160	.31791741
.1600	.00060575	.00060747	.00000172	.28409457
.1700	.00072122	.00072305	.00000183	.25347500
.1800	.00084977	.00085176	.00000198	.23358148
.1900	.00099194	.00099415	.00000221	.22277445
.2000	.00114852	.00115076	.00000223	.19429285
.2100	.00131972	.00132210	.00000239	.18091771
.2200	.00150611	.00150869	.00000258	.17150199
.2300	.00170826	.00171099	.00000273	.15988236
.2400	.00192663	.00192946	.00000283	.14707569
.2500	.00216158	.00216455	.00000297	.13758949
.2600	.00241343	.00241668	.00000325	.13468625
.2700	.00268285	.00268625	.00000340	.12669754
.2800	.00297001	.00297365	.00000364	.12256882
.2900	.00327547	.00327925	.00000378	.11546147
.3000	.00359950	.00360340	.00000391	.10854294
.3100	.00394229	.00394645	.00000416	.10562133
.3200	.00430447	.00430872	.00000425	.09880199
.3300	.00468595	.00469051	.00000456	.09731220
.3400	.00508744	.00509211	.00000467	.09174353
.3500	.00550891	.00551380	.00000489	.08873983
.3600	.00595077	.00595584	.00000507	.08525786
.3700	.00641321	.00641849	.00000528	.08226300
.3800	.00689645	.00690196	.00000551	.07987584
.3900	.00740072	.00740649	.00000577	.07795727
.4000	.00792635	.00793228	.00000593	.07478345
.4100	.00847337	.00847953	.00000615	.07262633
.4200	.00904200	.00904841	.00000641	.07091795
.4300	.00963240	.00963910	.00000670	.06960364
.4400	.01024484	.01025176	.00000692	.06751689
.4500	.01087936	.01088653	.00000717	.06593221
.4600	.01153610	.01154354	.00000745	.06453656
.4700	.01221512	.01222292	.00000781	.06391262
.4800	.01291670	.01292479	.00000809	.06260113
.4900	.01364084	.01364923	.00000839	.06153582
.5000	.01438771	.01439635	.00000864	.06001648

Table 7. (cont.)

TIME	EXACT	APPROX.	ERROR	REL. ERROR
.5100	.01515719	.01516622	.00000904	.05962189
.5200	.01594962	.01595892	.00000931	.05833995
.5300	.01676477	.01677451	.00000975	.05814576
.5400	.01760304	.01761305	.00001001	.05684245
.5500	.01846414	.01847456	.00001042	.05645538
.5600	.01934822	.01935910	.00001088	.05624290
.5700	.02025548	.02026669	.00001121	.05532330
.5800	.02118561	.02119733	.00001172	.05533945
.5900	.02213904	.02215106	.00001202	.05427968
.6000	.02311530	.02312786	.00001256	.05432333
.6100	.02411462	.02412773	.00001311	.05435707
.6200	.02513712	.02515066	.00001354	.05385661
.6300	.02618250	.02619663	.00001413	.05395207
.6400	.02725094	.02726560	.00001466	.05381099
.6500	.02834237	.02835756	.00001519	.05358762
.6600	.02945660	.02947245	.00001585	.05379440
.6700	.03059376	.03061022	.00001646	.05380182
.6800	.03175359	.03177083	.00001724	.05428048
.6900	.03293631	.03295420	.00001789	.05432910
.7000	.03414156	.03416029	.00001873	.05485104
.7100	.03536949	.03538901	.00001952	.05518315
.7200	.03661997	.03664029	.00002032	.05549158
.7300	.03789289	.03791406	.00002117	.05586272
.7400	.03918798	.03921021	.00002223	.05673168
.7500	.04050548	.04052865	.00002317	.05720461
.7600	.04184503	.04186927	.00002424	.05792803
.7700	.04320670	.04323197	.00002527	.05847473
.7800	.04459019	.04461663	.00002644	.05929331
.7900	.04599554	.04602315	.00002761	.06002756
.8000	.04742263	.04745142	.00002879	.06070098
.8100	.04887127	.04890131	.00003004	.06146351
.8200	.05034132	.05037269	.00003137	.06232058
.8300	.05183271	.05186545	.00003274	.06316089
.8400	.05334517	.05337945	.00003428	.06425137
.8500	.05487873	.05491455	.00003582	.06526572
.8600	.05643315	.05647062	.00003747	.06639006
.8700	.05800840	.05804751	.00003911	.06741610
.8800	.05960417	.05964507	.00004090	.06861936
.8900	.06122044	.06126315	.00004271	.06976592
.9000	.06285697	.06290160	.00004463	.07100088
.9100	.06451356	.06456026	.00004670	.07239098
.9200	.06619019	.06623898	.00004879	.07371032
.9300	.06788658	.06793759	.00005101	.07513414
.9400	.06960264	.06965592	.00005328	.07654307
.9500	.07133810	.07139380	.00005570	.07808310
.9600	.07309287	.07315108	.00005821	.07963841
.9700	.07486682	.07492758	.00006076	.08115210
.9800	.07665963	.07672312	.00006349	.08282065
.9900	.07847121	.07853753	.00006632	.08451635

ERROR LC-2 IN STATEMENT 0001 + 00 LINES

Table 8.

SOLUTION OF D(3)Y/DT(3)+3D(2)Y/DT(2)+3DY/DT+Y=1,MOD. SIMP. CONV.

TIME	EXACT	APPROX.	ERROR	REL. ERROR
.0040	-.00000010	.00000001	.00000011	-110.91657000
.0050	-.00000000	.00000002	.00000002	
.0060	-.00000000	.00000004	.00000004	
.0070	-.00000000	.00000006	.00000006	
.0080	-.00000000	.00000009	.00000009	
.0090	.00000010	.00000012	.00000002	22.34238000
.0100	.00000010	.00000017	.00000007	68.79092400
.0110	.00000020	.00000022	.00000002	11.10989600
.0120	.00000020	.00000029	.00000009	45.52710000
.0130	.00000030	.00000036	.00000007	22.71997600
.0140	.00000040	.00000045	.00000006	14.45889700
.0150	.00000049	.00000056	.00000006	12.64812900
.0160	.00000059	.00000068	.00000008	13.84765100
.0170	.00000079	.00000081	.00000002	2.16957980
.0180	.00000089	.00000096	.00000007	7.93652320
.0190	.00000109	.00000113	.00000004	3.72433190
.0200	.00000129	.00000132	.00000003	2.25215740
.0210	.00000148	.00000152	.00000004	2.56267150
.0220	.00000168	.00000175	.00000007	4.09320720
.0230	.00000198	.00000200	.00000002	.96763256
.0240	.00000227	.00000227	.00000000	.29443459
.0250	.00000257	.00000256	.00000000	.37543668
.0260	.00000286	.00000288	.00000001	.50181423
.0270	.00000326	.00000322	.00000004	1.21358990
.0280	.00000355	.00000359	.00000004	1.08279460
.0290	.00000394	.00000398	.00000004	1.03136060
.0300	.00000443	.00000440	.00000003	.67516227
.0310	.00000482	.00000485	.00000003	.68225284
.0320	.00000541	.00000533	.00000008	1.45832650
.0330	.00000580	.00000585	.00000004	.77213281
.0340	.00000639	.00000639	.00000000	.02987950
.0350	.00000687	.00000696	.00000009	1.31191130
.0360	.00000756	.00000757	.00000001	.14505638
.0370	.00000825	.00000821	.00000003	.37992258
.0380	.00000893	.00000889	.00000004	.40966417
.0390	.00000961	.00000960	.00000000	.02776367
.0400	.00001029	.00001035	.00000007	.64553346
.0410	.00001117	.00001114	.00000003	.23463081
.0420	.00001194	.00001197	.00000003	.21556102
.0430	.00001282	.00001283	.00000002	.13443251
.0440	.00001369	.00001374	.00000005	.34595531
.0450	.00001466	.00001469	.00000002	.15942999
.0460	.00001563	.00001568	.00000005	.29029782
.0470	.00001670	.00001671	.00000000	.03160006
.0480	.00001777	.00001778	.00000002	.09930878
.0490	.00001893	.00001890	.00000002	.13085031
.0500	.00001999	.00002007	.00000008	.41228092

Table 8. (cont.)

TIME	EXACT	APPROX.	ERROR	REL. ERROR
.0510	.00002125	.00002128	.00000004	.16526487
.0520	.00002250	.00002254	.00000004	.16270668
.0530	.00002386	.00002385	.00000000	.03746039
.0540	.00002521	.00002520	.00000000	.02181277
.0550	.00002666	.00002661	.00000005	.17528097
.0560	.00002800	.00002807	.00000007	.24474883
.0570	.00002964	.00002957	.00000007	.23244580
.0580	.00003108	.00003114	.00000006	.18217037
.0590	.00003272	.00003275	.00000003	.09696853
.0600	.00003445	.00003442	.00000003	.09883657
.0610	.00003618	.00003614	.00000004	.11764966
.0620	.00003790	.00003792	.00000001	.03244070
.0630	.00003973	.00003975	.00000002	.05954347
.0640	.00004164	.00004164	.00000000	.00200290
.0650	.00004366	.00004359	.00000007	.15862745
.0660	.00004567	.00004560	.00000007	.15590040
.0670	.00004767	.00004767	.00000000	.01213101
.0680	.00004988	.00004979	.00000008	.16679899
.0690	.00005197	.00005198	.00000002	.02977432
.0700	.00005426	.00005423	.00000003	.05229162
.0710	.00005655	.00005655	.00000000	.00927074
.0720	.00005894	.00005893	.00000000	.01634463
.0730	.00006142	.00006137	.00000005	.07448608
.0740	.00006389	.00006388	.00000001	.02080578
.0750	.00006646	.00006645	.00000000	.00720473
.0760	.00006912	.00006909	.00000003	.03666939
.0770	.00007178	.00007180	.00000003	.03526422
.0780	.00007463	.00007458	.00000005	.06932149
.0790	.00007748	.00007742	.00000006	.07112563
.0800	.00008032	.00008034	.00000002	.02947376
.0810	.00008335	.00008333	.00000002	.02702155
.0820	.00008638	.00008638	.00000000	.00981645
.0830	.00008961	.00008951	.00000009	.10185089
.0840	.00009271	.00009272	.00000000	.00475445
.0850	.00009603	.00009599	.00000003	.03169585
.0860	.00009943	.00009935	.00000008	.08442388
.0870	.00010282	.00010277	.00000005	.04751888
.0880	.00010631	.00010628	.00000004	.03605331
.0890	.00010989	.00010986	.00000004	.03421482
.0900	.00011357	.00011351	.00000005	.04554204
.0910	.00011723	.00011725	.00000002	.01667665
.0920	.00012119	.00012106	.00000013	.10796659
.0930	.00012505	.00012496	.00000009	.06964659
.0940	.00012899	.00012893	.00000005	.04197303
.0950	.00013312	.00013299	.00000013	.09898484
.0960	.00013725	.00013713	.00000012	.08659595
.0970	.00014146	.00014135	.00000011	.08126478
.0980	.00014577	.00014565	.00000012	.08223768
.0990	.00015017	.00015004	.00000013	.08688733

ERROR LC-2 IN STATEMENT 0001 + 00 LINES

Table 9.

SOLUTION OF D(3)Y/DT(3)+3D(2)Y/DT(2)+3DY/DT+Y=1,TRAP. CONV.

TIME	EXACT	APPROX.	ERROR	REL. ERROR
.0300	.00000443	.00000463	.00000020	4.42164560
.0400	.00001029	.00001065	.00000036	3.48976860
.0500	.00001999	.00002043	.00000044	2.19696700
.0600	.00003445	.00003484	.00000039	1.12106020
.0700	.00005426	.00005472	.00000046	.84381070
.0800	.00008032	.00008090	.00000058	.72564075
.0900	.00011357	.00011416	.00000059	.52331080
.1000	.00015467	.00015527	.00000060	.38869256
.1100	.00020434	.00020497	.00000063	.30825102
.1200	.00026331	.00026398	.00000067	.25442393
.1300	.00033217	.00033299	.00000082	.24640172
.1400	.00041183	.00041267	.00000084	.20412618
.1500	.00050284	.00050366	.00000083	.16441974
.1600	.00060575	.00060659	.00000084	.13925240
.1700	.00072122	.00072206	.00000084	.11646467
.1800	.00084977	.00085065	.00000088	.10366781
.1900	.00099194	.00099292	.00000099	.09930883
.2000	.00114852	.00114941	.00000088	.07673327
.2100	.00131972	.00132062	.00000091	.06875722
.2200	.00150611	.00150707	.00000097	.06429167
.2300	.00170826	.00170923	.00000098	.05722210
.2400	.00192663	.00192756	.00000094	.04860827
.2500	.00216158	.00216250	.00000093	.04298717
.2600	.00241343	.00241448	.00000105	.04364751
.2700	.00268285	.00268389	.00000105	.03895117
.2800	.00297001	.00297113	.00000112	.03786862
.2900	.00327547	.00327657	.00000110	.03358607
.3000	.00359950	.00360055	.00000105	.02928743
.3100	.00394229	.00394343	.00000114	.02880051
.3200	.00430447	.00430551	.00000104	.02427478
.3300	.00468595	.00468712	.00000117	.02492772
.3400	.00508744	.00508853	.00000109	.02136633
.3500	.00550891	.00551003	.00000111	.02023630
.3600	.00595077	.00595187	.00000110	.01850517
.3700	.00641321	.00641431	.00000110	.01714274
.3800	.00689645	.00689758	.00000112	.01628083
.3900	.00740072	.00740189	.00000117	.01577954
.4000	.00792635	.00792746	.00000110	.01391434
.4100	.00847337	.00847447	.00000110	.01295942
.4200	.00904200	.00904312	.00000112	.01235125
.4300	.00963240	.00963356	.00000116	.01204996
.4400	.01024484	.01024596	.00000112	.01090305
.4500	.01087936	.01088047	.00000111	.01020281
.4600	.01153610	.01153721	.00000111	.00963064
.4700	.01221512	.01221631	.00000119	.00977477
.4800	.01291670	.01291789	.00000119	.00918191
.4900	.01364084	.01364203	.00000120	.00877512
.5000	.01438771	.01438884	.00000113	.00785392

Table 9. (cont.)

TIME	EXACT	APPROX.	ERROR	REL. ERROR
.5100	.01515719	.01515840	.00000121	.00800280
.5200	.01594962	.01595077	.00000115	.00721020
.5300	.01676477	.01676602	.00000125	.00745015
.5400	.01760304	.01760419	.00000115	.00652728
.5500	.01846414	.01846533	.00000119	.00646659
.5600	.01934822	.01934948	.00000126	.00653290
.5700	.02025548	.02025666	.00000118	.00584533
.5800	.02118561	.02118689	.00000128	.00604656
.5900	.02213904	.02214017	.00000113	.00512217
.6000	.02311530	.02311651	.00000121	.00524761
.6100	.02411462	.02411590	.00000128	.00531213
.6200	.02513712	.02513832	.00000120	.00478575
.6300	.02618250	.02618376	.00000126	.00480092
.6400	.02725094	.02725217	.00000123	.00452461
.6500	.02834237	.02834353	.00000116	.00410693
.6600	.02945660	.02945780	.00000120	.00406700
.6700	.03059376	.03059492	.00000116	.00377528
.6800	.03175359	.03175483	.00000124	.00390507
.6900	.03293631	.03293748	.00000117	.00355535
.7000	.03414156	.03414280	.00000124	.00363194
.7100	.03536949	.03537072	.00000123	.00346909
.7200	.03661997	.03662115	.00000118	.00322775
.7300	.03789289	.03789402	.00000113	.00297681
.7400	.03918798	.03918923	.00000125	.00318465
.7500	.04050548	.04050670	.00000122	.00299959
.7600	.04184503	.04184631	.00000128	.00306608
.7700	.04320670	.04320798	.00000128	.00296250
.7800	.04459019	.04459159	.00000140	.00314195
.7900	.04599554	.04599703	.00000149	.00324597
.8000	.04742263	.04742419	.00000156	.00328324
.8100	.04887127	.04887292	.00000165	.00338236
.8200	.05034132	.05034312	.00000180	.00357758
.8300	.05183271	.05183464	.00000193	.00373123
.8400	.05334517	.05334736	.00000219	.00410346
.8500	.05487873	.05488113	.00000240	.00436781
.8600	.05643315	.05643581	.00000266	.00470823
.8700	.05800840	.05801125	.00000285	.00491308
.8800	.05960417	.05960731	.00000314	.00526809
.8900	.06122044	.06122383	.00000339	.00554227
.9000	.06285697	.06286066	.00000369	.00587206
.9100	.06451356	.06451764	.00000408	.00631650
.9200	.06619019	.06619459	.00000440	.00664298
.9300	.06788658	.06789135	.00000477	.00702790
.9400	.06960264	.06960775	.00000511	.00734742
.9500	.07133810	.07134362	.00000552	.00774341
.9600	.07309287	.07309879	.00000592	.00809792
.9700	.07486682	.07487307	.00000625	.00835083
.9800	.07665963	.07666630	.00000667	.00869558
.9900	.07847121	.07847827	.00000706	.00900075

ERROR LC-2 IN STATEMENT 0001 + 00 LINES

Table 10.

SOLUTION OF D(3)Y/DT(3)+3D(2)Y/DT(2)+3DY/DT+Y=1,TRAP. CONV.

TIME	EXACT	APPROX.	ERROR	REL. ERROR
.0030	-.00000010	.00000000	.00000011	-104.72249000
.0040	-.00000010	.00000001	.00000011	-110.91334000
.0050	-.00000000	.00000002	.00000002	
.0060	-.00000000	.00000004	.00000004	
.0070	-.00000000	.00000006	.00000006	
.0080	-.00000000	.00000009	.00000009	
.0090	.00000010	.00000012	.00000002	22.31302400
.0100	.00000010	.00000017	.00000007	68.75311900
.0110	.00000020	.00000022	.00000002	11.08661700
.0120	.00000020	.00000029	.00000009	45.49851100
.0130	.00000030	.00000036	.00000007	22.69731600
.0140	.00000040	.00000045	.00000006	14.43896200
.0150	.00000049	.00000056	.00000006	12.62958800
.0160	.00000059	.00000068	.00000008	13.82990400
.0170	.00000079	.00000081	.00000002	2.15446370
.0180	.00000089	.00000096	.00000007	7.92134280
.0190	.00000109	.00000113	.00000004	3.71044550
.0200	.00000129	.00000132	.00000003	2.23911960
.0210	.00000148	.00000152	.00000004	2.55020530
.0220	.00000168	.00000175	.00000007	4.08114530
.0230	.00000198	.00000199	.00000002	.95648918
.0240	.00000227	.00000226	.00000000	.30489807
.0250	.00000257	.00000256	.00000000	.38535237
.0260	.00000286	.00000287	.00000001	.49235507
.0270	.00000326	.00000322	.00000004	1.22735300
.0280	.00000355	.00000358	.00000004	1.07436710
.0290	.00000394	.00000398	.00000004	1.02347600
.0300	.00000443	.00000440	.00000003	.68238169
.0310	.00000482	.00000485	.00000003	.67549689
.0320	.00000541	.00000533	.00000008	1.46435570
.0330	.00000580	.00000585	.00000004	.76658743
.0340	.00000639	.00000639	.00000000	.03473345
.0350	.00000687	.00000696	.00000009	1.30767190
.0360	.00000756	.00000757	.00000001	.14157142
.0370	.00000825	.00000821	.00000003	.38265613
.0380	.00000893	.00000889	.00000004	.41163880
.0390	.00000961	.00000960	.00000000	.02896065
.0400	.00001029	.00001035	.00000007	.64514467
.0410	.00001117	.00001114	.00000003	.23417417
.0420	.00001194	.00001197	.00000003	.21690074
.0430	.00001282	.00001283	.00000002	.13668736
.0440	.00001369	.00001374	.00000005	.34916144
.0450	.00001466	.00001469	.00000002	.16359680
.0460	.00001563	.00001568	.00000005	.29545440
.0470	.00001670	.00001671	.00000000	.03774884
.0480	.00001777	.00001778	.00000002	.10648549
.0490	.00001893	.00001891	.00000002	.12265103

Table 10. (cont.)

TIME	EXACT	APPROX.	ERROR	REL. ERROR
.0500	.00001999	.00002007	.00000008	.42159193
.0510	.00002125	.00002128	.00000004	.17563834
.0520	.00002250	.00002254	.00000004	.17418449
.0530	.00002386	.00002385	.00000000	.02488138
.0540	.00002521	.00002521	.00000000	.00808409
.0550	.00002666	.00002661	.00000004	.16040694
.0560	.00002800	.00002807	.00000007	.26088540
.0570	.00002964	.00002958	.00000006	.21516400
.0580	.00003108	.00003114	.00000006	.20077803
.0590	.00003272	.00003276	.00000004	.11684211
.0600	.00003445	.00003442	.00000003	.07769276
.0610	.00003618	.00003615	.00000003	.09519269
.0620	.00003790	.00003792	.00000002	.05626995
.0630	.00003973	.00003976	.00000003	.08475895
.0640	.00004164	.00004165	.00000001	.02860979
.0650	.00004366	.00004360	.00000006	.13060567
.0660	.00004567	.00004561	.00000006	.12637703
.0670	.00004767	.00004768	.00000000	.01897582
.0680	.00004988	.00004981	.00000007	.13416859
.0690	.00005197	.00005200	.00000003	.06409186
.0700	.00005426	.00005425	.00000000	.01633710
.0710	.00005655	.00005657	.00000002	.02841517
.0720	.00005894	.00005895	.00000001	.02310950
.0730	.00006142	.00006140	.00000002	.03323103
.0740	.00006389	.00006391	.00000001	.02234434
.0750	.00006646	.00006648	.00000003	.03786248
.0760	.00006912	.00006912	.00000000	.01033167
.0770	.00007178	.00007184	.00000006	.08429641
.0780	.00007463	.00007462	.00000001	.01830787
.0790	.00007748	.00007746	.00000001	.01803470
.0800	.00008032	.00008038	.00000007	.08474703
.0810	.00008335	.00008337	.00000003	.03039534
.0820	.00008638	.00008644	.00000006	.06947233
.0830	.00008961	.00008957	.00000004	.04000201
.0840	.00009271	.00009278	.00000006	.06897400
.0850	.00009603	.00009606	.00000003	.03484085
.0860	.00009943	.00009941	.00000002	.01553851
.0870	.00010282	.00010285	.00000002	.02381779
.0880	.00010631	.00010635	.00000004	.03775580
.0890	.00010989	.00010994	.00000005	.04209515
.0900	.00011357	.00011360	.00000004	.03329359
.0910	.00011723	.00011734	.00000012	.09813205
.0920	.00012119	.00012117	.00000003	.02400266
.0930	.00012505	.00012507	.00000002	.01701779
.0940	.00012899	.00012905	.00000006	.04744642
.0950	.00013312	.00013311	.00000000	.00682084
.0960	.00013725	.00013726	.00000001	.00843737
.0970	.00014146	.00014149	.00000002	.01670396
.0980	.00014577	.00014580	.00000003	.01873466
.0990	.00015017	.00015020	.00000003	.01714706

ERROR LC-2 IN STATEMENT 0001 + 00 LINES

Table 11.

SOLUTION OF DY/DT+Y*Y=1,MODIFIED SIMPSONS CONVOLUTION

TIME	EXACT	APPROX.	ERROR	REL. ERROR
.060	.05992810	.05992200	.00000610	.01017886
.090	.08975780	.08974500	.00001280	.01426060
.120	.11942730	.11941100	.00001630	.01364847
.150	.14888500	.14886500	.00002000	.01343319
.180	.17808090	.17805700	.00002390	.01342087
.210	.20696650	.20693800	.00002850	.01377034
.240	.23549580	.23546300	.00003280	.01392806
.270	.26362490	.26358500	.00003990	.01513514
.300	.29131260	.29126800	.00004460	.01531001
.330	.31852080	.31847000	.00005080	.01594872
.360	.34521410	.34515700	.00005710	.01654046
.390	.37136030	.37130200	.00005830	.01569904
.420	.39693050	.39686900	.00006150	.01549390
.450	.42189900	.42183400	.00006500	.01540653
.480	.44624360	.44617500	.00006860	.01537277
.510	.46994520	.46987500	.00007020	.01493791
.540	.49298800	.49291700	.00007100	.01440197
.570	.51535930	.51528800	.00007130	.01383501
.600	.53704960	.53697800	.00007160	.01333210
.630	.55805230	.55797700	.00007530	.01349336
.660	.57836350	.57828600	.00007750	.01339988
.690	.59798200	.59790200	.00008000	.01337833
.720	.61690930	.61682900	.00008030	.01301650
.750	.63514900	.63506900	.00008000	.01259547
.780	.65270670	.65262800	.00007870	.01205748
.810	.66959030	.66951100	.00007930	.01184306
.840	.68580910	.68572800	.00008110	.01182545
.870	.70137420	.70129400	.00008020	.01143470
.900	.71629790	.71621700	.00008090	.01129418
.930	.73059390	.73051200	.00008190	.01121006
.960	.74427690	.74419500	.00008190	.01100397
.990	.75736240	.75728000	.00008240	.01087986
1.020	.76986660	.76978600	.00008060	.01046935
1.050	.78180640	.78172500	.00008140	.01041179
1.080	.79319910	.79311700	.00008210	.01035049
1.110	.80406240	.80398200	.00008040	.00999922
1.140	.81441410	.81433500	.00007910	.00971250
1.170	.82427220	.82419400	.00007820	.00948716
1.200	.83365460	.83357600	.00007860	.00942837
1.230	.84257940	.84250300	.00007640	.00906739
1.260	.85106410	.85098700	.00007710	.00905925
1.290	.85912660	.85905200	.00007460	.00868324
1.320	.86678400	.86670900	.00007500	.00865267
1.350	.87405330	.87398000	.00007330	.00838622
1.380	.88095130	.88087600	.00007530	.00854758
1.410	.88749420	.88741800	.00007620	.00858597

Table 11. (cont.)

TIME	EXACT	APPROX.	ERROR	REL. ERROR
1.440	.89369780	.89362100	.00007680	.00859351
1.470	.89957750	.89950200	.00007550	.00839283
1.500	.90514830	.90507200	.00007630	.00842956
1.530	.91042460	.91035000	.00007460	.00819398
1.560	.91542050	.91534500	.00007550	.00824758
1.590	.92014940	.92007500	.00007440	.00808564
1.620	.92462430	.92455200	.00007230	.00781939
1.650	.92885770	.92878800	.00006970	.00750384
1.680	.93286160	.93279100	.00007060	.00756811
1.710	.93664760	.93657800	.00006960	.00743076
1.740	.94022670	.94015500	.00007170	.00762582
1.770	.94360950	.94353800	.00007150	.00757729
1.800	.94680610	.94673500	.00007110	.00750946
1.830	.94982610	.94975700	.00006910	.00727502
1.860	.95267890	.95261100	.00006790	.00712727
1.890	.95537320	.95530600	.00006720	.00703390
1.920	.95791740	.95784900	.00006840	.00714049
1.950	.96031940	.96025000	.00006940	.00722676
1.980	.96258700	.96251900	.00006800	.00706430
2.010	.96472740	.96466100	.00006640	.00688277
2.040	.96674730	.96668200	.00006530	.00675461
2.070	.96865350	.96859000	.00006350	.00655549
2.100	.97045200	.97038900	.00006300	.00649182
2.130	.97214880	.97208500	.00006380	.00656278
2.160	.97374940	.97368700	.00006240	.00640822
2.190	.97525920	.97519800	.00006120	.00627525
2.220	.97668320	.97662000	.00006320	.00647088
2.250	.97802620	.97796500	.00006120	.00625750
2.280	.97929260	.97923100	.00006160	.00629025
2.310	.98048670	.98042800	.00005870	.00598682
2.340	.98161260	.98155400	.00005860	.00596977
2.370	.98267420	.98261600	.00005820	.00592261
2.400	.98367490	.98361600	.00005890	.00598775
2.430	.98461830	.98455900	.00005930	.00602264
2.460	.98550760	.98544800	.00005960	.00604764
2.490	.98634580	.98628900	.00005680	.00575863
2.520	.98713580	.98708000	.00005580	.00565272
2.550	.98788040	.98782500	.00005540	.00560797
2.580	.98858220	.98852600	.00005620	.00568491
2.610	.98924360	.98918600	.00005760	.00582263
2.640	.98986680	.98980900	.00005780	.00583917
2.670	.99045410	.99039500	.00005910	.00596696
2.700	.99100750	.99094800	.00005950	.00600399
2.730	.99152900	.99146900	.00006000	.00605126
2.760	.99202030	.99196100	.00005930	.00597770
2.790	.99248330	.99242500	.00005830	.00587415
2.820	.99291950	.99286300	.00005650	.00569029
2.850	.99333040	.99327300	.00005740	.00577854
2.880	.99371760	.99366200	.00005560	.00559515
2.910	.99408240	.99402700	.00005540	.00557298
2.940	.99442610	.99437200	.00005410	.00544032
2.970	.99474980	.99469600	.00005380	.00540840

ERROR LC-2 IN STATEMENT 0001 + 00 LINES

Table 12.

SOLUTION OF DY/DT+Y*Y=1,MODIFIED SIMPSONS CONVOLUTION

TIME	EXACT	APPROX.	ERROR	REL. ERROR
.002	.00200000	.00190000	.00010000	5.00000000
.003	.00300000	.00250000	.00050000	16.66666700
.004	.00400000	.00310000	.00090000	22.50000000
.005	.00500000	.00370000	.00130000	26.00000000
.006	.00599990	.00430000	.00169990	28.33213900
.007	.00699990	.00490000	.00209990	29.99900000
.008	.00799980	.00550000	.00249980	31.24828100
.009	.00899980	.00610000	.00289980	32.22071600
.010	.00999970	.00670000	.00329970	32.99799000
.011	.01099960	.00730000	.00369960	33.63395000
.012	.01199950	.00790000	.00409950	34.16392300
.013	.01299930	.00850000	.00449930	34.61186400
.014	.01399910	.00910000	.00489910	34.99582100
.015	.01499890	.00970000	.00529890	35.32859100
.016	.01599870	.01030000	.00569870	35.61976900
.017	.01699840	.01090000	.00609840	35.87631800
.018	.01799810	.01150000	.00649810	36.10436700
.019	.01899770	.01210000	.00689770	36.30807900
.020	.01999730	.01270000	.00729730	36.49142600
.021	.02099690	.01330000	.00769690	36.65731600
.022	.02199650	.01390000	.00809650	36.80812900
.023	.02299590	.01450000	.00849590	36.94528200
.024	.02399540	.01510000	.00889540	37.07127200
.025	.02499480	.01570000	.00929480	37.18693500
.026	.02599420	.01630000	.00969420	37.29370400
.027	.02699350	.01690000	.01009350	37.39233500
.028	.02799270	.01750000	.01049270	37.48370100
.029	.02899190	.01810000	.01089190	37.56876900
.030	.02999100	.01870000	.01129100	37.64796100
.031	.03099010	.01930000	.01169010	37.72204700
.032	.03198910	.01990000	.01208910	37.79131000
.033	.03298810	.02050000	.01248810	37.85637900
.034	.03398690	.02110000	.01288690	37.91725600
.035	.03498570	.02170000	.01328570	37.97465800
.036	.03598450	.02230000	.01368450	38.02887400
.037	.03698320	.02290000	.01408320	38.07999300
.038	.03798170	.02350000	.01448170	38.12809900
.039	.03898030	.02410000	.01488030	38.17389800
.040	.03997870	.02470000	.01527870	38.21710100
.041	.04097700	.02530000	.01567700	38.25804700
.042	.04197530	.02590000	.01607530	38.29704600
.043	.04297350	.02650000	.01647350	38.33409000
.044	.04397170	.02710000	.01687170	38.36945100
.045	.04496970	.02770000	.01726970	38.40296900
.046	.04596760	.02830000	.01766760	38.43489800
.047	.04696540	.02890000	.01806540	38.46533800
.048	.04796320	.02950000	.01846320	38.49451200
.049	.04896080	.03010000	.01886080	38.52224600
.050	.04995840	.03070000	.01925840	38.54887300

Table 12. (cont.)

TIME	EXACT	APPROX.	ERROR	REL. ERROR
.051	.05095580	.03130000	.01965580	38.57421500
.052	.05195320	.03190000	.02005320	38.59858500
.053	.05295040	.03250000	.02045040	38.62180500
.054	.05394760	.03310000	.02084760	38.64416600
.055	.05494460	.03370000	.02124460	38.66549200
.056	.05594160	.03430000	.02164160	38.68605800
.057	.05693840	.03490000	.02203840	38.70568900
.058	.05793510	.03550000	.02243510	38.72453800
.059	.05893160	.03610000	.02283160	38.74254200
.060	.05992810	.03670000	.02322810	38.75994700
.061	.06092450	.03730000	.02362450	38.77668300
.062	.06192070	.03790000	.02402070	38.79268200
.063	.06291680	.03850000	.02441680	38.80807700
.064	.06391280	.03910000	.02481280	38.82289600
.065	.06490860	.03970000	.02520860	38.83707200
.066	.06590440	.04030000	.02560440	38.85082000
.067	.06690000	.04090000	.02600000	38.86397600
.068	.06789540	.04150000	.02639540	38.87656600
.069	.06889070	.04210000	.02679070	38.88870300
.070	.06988590	.04270000	.02718590	38.90040800
.071	.07088100	.04330000	.02758100	38.91169700
.072	.07187580	.04390000	.02797580	38.92241900
.073	.07287060	.04450000	.02837060	38.93284800
.074	.07386520	.04510000	.02876520	38.94283100
.075	.07485970	.04570000	.02915970	38.95246700
.076	.07585400	.04630000	.02955400	38.96169000
.077	.07684820	.04690000	.02994820	38.97059400
.078	.07784220	.04750000	.03034220	38.97911400
.079	.07883610	.04810000	.03073610	38.98734200
.080	.07982980	.04870000	.03112980	38.99521200
.081	.08082330	.04930000	.03152330	39.00273800
.082	.08181670	.04990000	.03191670	39.01000700
.083	.08280990	.05050000	.03230990	39.01695300
.084	.08380300	.05110000	.03270300	39.02366300
.085	.08479590	.05170000	.03309590	39.03007100
.086	.08578860	.05230000	.03348860	39.03618900
.087	.08678120	.05290000	.03388120	39.04209700
.088	.08777360	.05350000	.03427360	39.04773200
.089	.08876570	.05410000	.03466570	39.05303500
.090	.08975780	.05470000	.03505780	39.05822100
.091	.09074960	.05530000	.03544960	39.06309200
.092	.09174130	.05590000	.03584130	39.06779200
.093	.09273280	.05650000	.03623280	39.07225900
.094	.09372420	.05710000	.03662420	39.07656700
.095	.09471520	.05770000	.03701520	39.08052800
.096	.09570620	.05830000	.03740620	39.08440600
.097	.09669690	.05890000	.03779690	39.08801600
.098	.09768750	.05950000	.03818750	39.09149100
.099	.09867780	.06010000	.03857780	39.09471000

ERROR LC-2 IN STATEMENT 0001 + 00 LINES

Table 13.

C C SOLUTION OF DY/DT+Y*Y=1,TRAPEZOIDAL CONVOLUTION

TIME	EXACT	APPROX.	ERROR	REL. ERROR
.030	.02999100	.02998600	.00000500	.01667167
.060	.05992810	.05991600	.00001210	.02019086
.090	.08975780	.08974300	.00001480	.01648882
.120	.11942730	.11940600	.00002130	.01783512
.150	.14888500	.14886000	.00002500	.01679148
.180	.17808090	.17805300	.00002790	.01566704
.210	.20696650	.20693600	.00003050	.01473668
.240	.23549580	.23546000	.00003580	.01520197
.270	.26362490	.26358600	.00003890	.01475581
.300	.29131260	.29127100	.00004160	.01428019
.330	.31852080	.31847800	.00004280	.01343711
.360	.34521410	.34517100	.00004310	.01248501
.390	.37136030	.37131500	.00004530	.01219840
.420	.39693050	.39688700	.00004350	.01095910
.450	.42189900	.42185700	.00004200	.00995499
.480	.44624360	.44620300	.00004060	.00909817
.510	.46994520	.46990700	.00003820	.00812861
.540	.49298800	.49295300	.00003500	.00709956
.570	.51535930	.51532600	.00003330	.00646151
.600	.53704960	.53701900	.00003060	.00569780
.630	.55805230	.55802600	.00002630	.00471282
.660	.57836350	.57834300	.00002050	.00354448
.690	.59798200	.59796600	.00001600	.00267567
.720	.61690930	.61689900	.00001030	.00166961
.750	.63514900	.63514400	.00000500	.00078722
.780	.65270670	.65270400	.00000270	.00041366
.810	.66959030	.66959200	.00000170	.00025389
.840	.68580910	.68581600	.00000690	.00100611
.870	.70137420	.70138300	.00000880	.00125468
.900	.71629790	.71631100	.00001310	.00182885
.930	.73059390	.73060800	.00001410	.00192994
.960	.74427690	.74429400	.00001710	.00229753
.990	.75736240	.75738200	.00001960	.00258793
1.020	.76986660	.76988900	.00002240	.00290960
1.050	.78180640	.78183400	.00002760	.00353029
1.080	.79319910	.79323100	.00003190	.00402169
1.110	.80406240	.80409900	.00003660	.00455189
1.140	.81441410	.81445700	.00004290	.00526759
1.170	.82427220	.82432000	.00004780	.00579906
1.200	.83365460	.83370500	.00005040	.00604567
1.230	.84257940	.84263400	.00005460	.00648010
1.260	.85106410	.85112100	.00005690	.00668575
1.290	.85912660	.85918600	.00005940	.00691400
1.320	.86678400	.86684500	.00006100	.00703751
1.350	.87405330	.87411800	.00006470	.00740229
1.380	.88095130	.88101800	.00006670	.00757136
1.410	.88749420	.88756300	.00006880	.00775216
1.440	.89369780	.89376800	.00007020	.00785500

Table 13. (cont.)

TIME	EXACT	APPROX.	ERROR	REL. ERROR
1.470	.89957750	.89964900	.00007150	.00794818
1.500	.90514830	.90522300	.00007470	.00825279
1.530	.91042460	.91050200	.00007740	.00850153
1.560	.91542050	.91550000	.00007950	.00868453
1.590	.92014940	.92022900	.00007960	.00865077
1.620	.92462430	.92470700	.00008270	.00894417
1.650	.92885770	.92894100	.00008330	.00896800
1.680	.93286160	.93294600	.00008440	.00904743
1.710	.93664760	.93673300	.00008540	.00911762
1.740	.94022670	.94031100	.00008430	.00896592
1.770	.94360950	.94369400	.00008450	.00895498
1.800	.94680610	.94689300	.00008690	.00917823
1.830	.94982610	.94991300	.00008690	.00914904
1.860	.95267890	.95276500	.00008610	.00903767
1.890	.95537320	.95546000	.00008680	.00908545
1.920	.95791740	.95800400	.00008660	.00904045
1.950	.96031940	.96040900	.00008960	.00933023
1.980	.96258700	.96267500	.00008800	.00914203
2.010	.96472740	.96481500	.00008760	.00908029
2.040	.96674730	.96683500	.00008770	.00907166
2.070	.96865350	.96874300	.00008950	.00923963
2.100	.97045200	.97054100	.00008900	.00917098
2.130	.97214880	.97223800	.00008920	.00917555
2.160	.97374940	.97383700	.00008760	.00899615
2.190	.97525920	.97534600	.00008680	.00890020
2.220	.97668320	.97677100	.00008780	.00898961
2.250	.97802620	.97811500	.00008880	.00907951
2.280	.97929260	.97938100	.00008840	.00902692
2.310	.98048670	.98057600	.00008930	.00910772
2.340	.98161260	.98170100	.00008840	.00900559
2.370	.98267420	.98276100	.00008680	.00883304
2.400	.98367490	.98376300	.00008810	.00895621
2.430	.98461830	.98470700	.00008870	.00900857
2.460	.98550760	.98559600	.00008840	.00897000
2.490	.98634580	.98643300	.00008720	.00884071
2.520	.98713580	.98722300	.00008720	.00883364
2.550	.98788040	.98796800	.00008760	.00886747
2.580	.98858220	.98867100	.00008880	.00898256
2.610	.98924360	.98933400	.00009040	.00913830
2.640	.98986680	.98995800	.00009120	.00921336
2.670	.99045410	.99054500	.00009090	.00917761
2.700	.99100750	.99109800	.00009050	.00913212
2.730	.99152900	.99161900	.00009000	.00907689
2.760	.99202030	.99211000	.00008970	.00904215
2.790	.99248330	.99257400	.00009070	.00913869
2.820	.99291950	.99301000	.00009050	.00911454
2.850	.99333040	.99342100	.00009060	.00912083
2.880	.99371760	.99380900	.00009140	.00919778
2.910	.99408240	.99417400	.00009160	.00921453
2.940	.99442610	.99451700	.00009090	.00914095
2.970	.99474980	.99483900	.00008920	.00896708

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Table 14.

C C SOLUTION OF DY/DT+Y*Y=1,TRAPEZOIDAL CONVOLUTION

TIME	EXACT	APPROX.	ERROR	REL. ERROR
.001	.00100000	.00120000	.00020000	20.00000000
.002	.00200000	.00220000	.00020000	10.00000000
.003	.00300000	.00320000	.00020000	6.66666670
.004	.00400000	.00420000	.00020000	5.00000000
.005	.00500000	.00520000	.00020000	4.00000000
.006	.00599990	.00620000	.00020010	3.33505560
.007	.00699990	.00720000	.00020010	2.85861230
.008	.00799980	.00820000	.00020020	2.50256260
.009	.00899980	.00920000	.00020020	2.22449390
.010	.00999970	.01020000	.00020030	2.00306010
.011	.01099960	.01120000	.00020040	1.82188440
.012	.01199950	.01220000	.00020050	1.67090300
.013	.01299930	.01320000	.00020070	1.54392930
.014	.01399910	.01420000	.00020090	1.43509230
.015	.01499890	.01520000	.00020110	1.34076500
.016	.01599870	.01620000	.00020130	1.25822720
.017	.01699840	.01720000	.00020160	1.18599400
.018	.01799810	.01820000	.00020190	1.12178510
.019	.01899770	.01920000	.00020230	1.06486570
.020	.01999730	.02020000	.00020270	1.01363680
.021	.02099690	.02120000	.00020310	.96728565
.022	.02199650	.02220000	.00020350	.92514718
.023	.02299590	.02320000	.00020410	.88754952
.024	.02399540	.02420000	.00020460	.85266343
.025	.02499480	.02520000	.00020520	.82097076
.026	.02599420	.02620000	.00020580	.79171507
.027	.02699350	.02720000	.00020650	.76499898
.028	.02799270	.02820000	.00020730	.74055021
.029	.02899190	.02920000	.00020810	.71778669
.030	.02999100	.03020000	.00020900	.69687573
.031	.03099010	.03120000	.00020990	.67731308
.032	.03198910	.03220000	.00021090	.65928707
.033	.03298810	.03320000	.00021190	.64235285
.034	.03398690	.03420000	.00021310	.62700629
.035	.03498570	.03520000	.00021430	.61253598
.036	.03598450	.03620000	.00021550	.59886896
.037	.03698320	.03720000	.00021680	.58621212
.038	.03798170	.03820000	.00021830	.57475047
.039	.03898030	.03920000	.00021970	.56361803
.040	.03997870	.04020000	.00022130	.55354476
.041	.04097700	.04120000	.00022300	.54420773
.042	.04197530	.04220000	.00022470	.53531482
.043	.04297350	.04320000	.00022650	.52706901
.044	.04397170	.04420000	.00022830	.51919757
.045	.04496970	.04520000	.00023030	.51212261
.046	.04596760	.04620000	.00023240	.50557349
.047	.04696540	.04720000	.00023460	.49951667
.048	.04796320	.04820000	.00023680	.49371185
.049	.04896080	.04920000	.00023920	.48855411
.050	.04995840	.05020000	.00024160	.48360236

Table 14. (cont.)

TIME	EXACT	APPROX.	ERROR	REL. ERROR
.051	.05095580	.05120000	.00024420	.47923887
.052	.05195320	.05220000	.00024680	.47504292
.053	.05295040	.05320000	.00024960	.47138454
.054	.05394760	.05420000	.00025240	.46786141
.055	.05494460	.05520000	.00025540	.46483185
.056	.05594160	.05620000	.00025840	.46191028
.057	.05693840	.05720000	.00026160	.45944389
.058	.05793510	.05820000	.00026490	.45723577
.059	.05893160	.05920000	.00026840	.45544326
.060	.05992810	.06020000	.00027190	.45371036
.061	.06092450	.06120000	.00027550	.45219903
.062	.06192070	.06220000	.00027930	.45106079
.063	.06291680	.06320000	.00028320	.45011825
.064	.06391280	.06420000	.00028720	.44936226
.065	.06490860	.06520000	.00029140	.44893897
.066	.06590440	.06620000	.00029560	.44852847
.067	.06690000	.06720000	.00030000	.44843049
.068	.06789540	.06820000	.00030460	.44863128
.069	.06889070	.06920000	.00030930	.44897207
.070	.06988590	.07020000	.00031410	.44944688
.071	.07088100	.07120000	.00031900	.45005008
.072	.07187580	.07220000	.00032420	.45105585
.073	.07287060	.07320000	.00032940	.45203415
.074	.07386520	.07420000	.00033480	.45325810
.075	.07485970	.07520000	.00034030	.45458371
.076	.07585400	.07620000	.00034600	.45613943
.077	.07684820	.07720000	.00035180	.45778561
.078	.07784220	.07820000	.00035780	.45964785
.079	.07883610	.07920000	.00036390	.46159057
.080	.07982980	.08020000	.00037020	.46373660
.081	.08082330	.08120000	.00037670	.46607847
.082	.08181670	.08220000	.00038330	.46848626
.083	.08280990	.08320000	.00039010	.47107894
.084	.08380300	.08420000	.00039700	.47373006
.085	.08479590	.08520000	.00040410	.47655606
.086	.08578860	.08620000	.00041140	.47955090
.087	.08678120	.08720000	.00041880	.48259300
.088	.08777360	.08820000	.00042640	.48579527
.089	.08876570	.08920000	.00043430	.48926556
.090	.08975780	.09020000	.00044220	.49265913
.091	.09074960	.09120000	.00045040	.49631073
.092	.09174130	.09220000	.00045870	.49999291
.093	.09273280	.09320000	.00046720	.50381311
.094	.09372420	.09420000	.00047580	.50765971
.095	.09471520	.09520000	.00048480	.51185026
.096	.09570620	.09620000	.00049380	.51595403
.097	.09669690	.09720000	.00050310	.52028555
.098	.09768750	.09820000	.00051250	.52463212
.099	.09867780	.09920000	.00052220	.52919704

ERROR LC-2 IN STATEMENT 0001 + 00 LINES

SUMMARY

Investigation of the graphs of the first order equation shows that there exists a sampling interval size which minimizes the largest absolute value of error for the fixed computation duration in Trapezoidal Convolution and the Modified Simpson's Convolution. Simpson's 1/3 rule shows a smaller absolute maximum error than the other two but it is oscillatory and does not tend to give a desired time interval which minimizes the maximum error.

In determining a sampling period which yields a minimum absolute maximum error it has been shown by Halijak (12), that T is a function of the solution time, t. This conclusion was arrived at using Trapezoidal Convolution and various differential equation solutions.

At about T = .008 seconds the Modified Simpson's Convolution starts and continues to give better results than Trapezoidal Convolution. Its oscillation gives a definite minimum at T = .026 seconds and at T = .052 the oscillation discontinues and the curve fits a pattern similar to that of Trapezoidal Convolution, but with better results.

The tendency of Simpson's formula to give oscillatory results has definitely been ameliorated by using the Modified Simpson's Convolution over Simpson's 1/3 rule; this can be seen from the graphs. There also tends to be a pattern when using the Modified Simpson's Convolution. The error curve seems to be fairly symmetrical on both sides of the desired T, of .026 seconds, and after a fashion, it tends to stair step down to, and up from, T = .026 seconds. This observation could possibly be useful in determining the absolute maximum error of other differential equations when using the same solution time.

The first order equation was the only one used in determining a stability analysis of the absolute maximum error, but the results of higher order equations should follow a similar pattern. The second order, third order and non-linear differential equations that were solved show only the decimal place accuracy that can be expected for various sampling periods.

In checking the second order equation for accuracy, sampling intervals of .0005 seconds, .001 seconds, and .01 seconds were used with a solution time of .1 second for the two smallest sampling intervals, and 1 second for the largest interval. Smaller intervals were not used because of the larger number of iterations required and thus much longer computer time. In comparing the data between Trapezoidal Convolution and the Modified Simpson's Convolution it was observed that both methods give very accurate results. With T = .01 seconds, the Modified Simpson's Convolution had superior accuracy for the first half of the solution time and Trapezoidal Convolution was superior the last half. Both methods were never below a five decimal place accuracy and for the first .5 seconds the accuracy was consistently six decimal places. With T = .001 both methods showed an increase in accuracy with Trapezoidal Convolution being consistent with seven decimal place accuracy and the Modified Simpson's Convolution having seven place accuracy except for the last .03 seconds which yielded six place accuracy. With T = .0005 seconds the overall accuracy of both methods decreased, with Trapezoidal Convolution showing a superior overall accuracy of six and seven decimal places.

The same sampling intervals, etc., were used for the third order differential equation and the accuracy was approximately the same as that obtained for the second order equation except the Modified Simpson's Convolution lost a decimal place accuracy for the first half of the solution time for T = .01 seconds.

The sampling intervals used for the non-linear equation were .001 seconds, .01 seconds, and .03 seconds. With T = .001 seconds, Trapezoidal Convolution shows superior accuracy by two decimal places, having three place accuracy. With T = .01 seconds and .03 seconds the accuracy of both methods was three and four decimal places, respectively.

For purpose of high decimal place accuracy and considering the complexity of the Modified Simpson's Convolution as compared to Trapezoidal Convolution, Trapezoidal Convolution appears to be superior for equations up to and including the third order. It should be noted that round-off error has not been taken into account in arriving at the preceding conclusions.

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APPENDICES

TABLE I

COEFFICIENTS $A(n,p)$ IN $A_n(z) = \sum_{p=-1}^{n-2} A(n,p)\, z^p$

FOR $Z(1/s^n) = \frac{T^{n-1} z}{(n-1)!} \frac{A_n(z)}{(1-z)^n}$

n \ P	-1	0	1	2	3	4	5	6	7	8	9	10	11	12	13
1	1														
2		1													
3		1	1												
4		1	4	1											
5		1	11	11	1										
6		1	26	66	26	1									
7		1	57	302	302	57	1								
8		1	120	1191	2416	1191	120	1							
9		1	247	4293	15619	15619	4293	247	1						
10		1	502	14,608	88,234	156,190	88234	14608	502	1					
11		1	1013	47,840	455,192	1,310,354	1,310,354	455192	47840	1013	1				
12		1	2036	152,637	2,203,488	9,738,114	15,724,248	9738114	2203488	152637	2036	1			
13		1	4083	478,271	10,187,685	66,318,474	162,512,286	162512286	66318474	10187685	478271	4083	1		
14		1	8178	1479726	45533450	423281535	1505621508	2275172004	1505621508	423281535	45533450	1479726	8178	1	
15		1	16369	4537314	198410786	2571742175	12843262863	27971176092	27971176092	12843262863	2571742175	198410786	4537314	16369	1

APPENDIX B

Solution of $\frac{dy}{dt} + y = 1 \qquad\qquad y(0) = 0$

Laplace transforming

$$s\bar{y} + \bar{y} = \frac{1}{s} \tag{77}$$

$$\bar{y} = \frac{1}{s(s+1)} = \frac{A}{s} + \frac{B}{s+1}, \qquad A = 1$$

$$B = -1$$

$$\bar{y} = \frac{1}{s} - \frac{1}{s+1}$$

Inverting into the time domain

$$y(t) = 1-e^{-t} \qquad \text{(exact solution)} \tag{78}$$

Trapezoidal Convolution:

From equation (77), dividing through by s

$$\bar{y} + \bar{\bar{y}} = \frac{1}{s^2} \tag{79}$$

Z transforming (79), using equations (54) and (55)

$$Z\bar{y} + \frac{T}{2}\,\frac{1+z}{1-z}Z\bar{y} = \frac{Tz}{(1-z)^2}$$

Multiplying through by 1-z

$$Z\bar{y}(1-z) + \frac{T}{2}(1+z)Z\bar{y} = TX_n \qquad \left\{X_n\right\} = \left\{0,\ 1,\ 1,\ 1,\ 1,\ \ldots\right\}$$

Collecting terms in powers of z

$$(1 + \tfrac{T}{2})Z\bar{y} + (\tfrac{T}{2} - 1)zZ\bar{y} = TX_n$$

From which the recurrence relation is

$$(1 + \tfrac{T}{2})y_n + (\tfrac{T}{2} - 1)y_{n-1} = TX_n \tag{80}$$

Modified Simpson's Convolution:

Z-transforming (79) using equations (55) and (57)

$$Z\bar{y} + \frac{T}{1-z}\left(\frac{5}{12} + \frac{2}{3}z - \frac{z^2}{12}\right)Z\bar{y} + \left(\frac{y_1}{4} - \frac{y_2}{12}\right)z = \frac{Tz}{(1-z)^2}$$

Multiplying through by 1-z:

$$Z\bar{y}(1-z) + T\left(\frac{5}{12} + \frac{2}{3}z - \frac{z^2}{12}\right)Z\bar{y} + T\left(\frac{y_1}{4} \quad \frac{y_2}{12}\right)z = TX_n$$

$$\left\{X_n\right\} = \left\{0,\ 1,\ 1,\ 1,\ \ldots\right\}$$

Collecting terms

$$\left(1 + \frac{5}{12}T\right)Z\bar{y} + \left(\frac{2}{3}T - 1\right)zZ\bar{y} - \frac{T}{12}z^2 Z\bar{y} = X_n$$

$$\left\{X_n\right\} = \left\{0, T\left(1 + \frac{y_2}{12} - \frac{y_1}{4}\right),\ T, T, T,\ \ldots\right\}$$

From which the recurrence relation is

$$\left(1 + \frac{5}{12}T\right)y_n + \left(\frac{2}{3}T - 1\right)y_{n-1} - \frac{T}{12}y_{n-2} = X_n \qquad (81)$$

Next y_1 and y_2 have to be determined;

From the recurrence relation

$$\left(1 + \frac{5}{12}T\right)y_1 = T + \frac{T}{12}y_2 - \frac{T}{4}y_1$$

$$\left(1 + \frac{2}{3}T\right)y_1 - \frac{T}{12}y_2 = T \qquad (82)$$

$$\left(\frac{2}{3}T - 1\right)y_1 + \left(1 + \frac{5}{12}T\right)y_2 = T \qquad (83)$$

Solving (82) and (83) for y_1 and y_2 we have

$$y_1 = \frac{\begin{bmatrix} T & -\frac{T}{12} \\ T & 1 + \frac{5}{12}T \end{bmatrix}}{\begin{bmatrix} 1 + \frac{2}{3}T & -\frac{T}{12} \\ \frac{2}{3}T & 1 + \frac{5}{12}T \end{bmatrix}} = \frac{T + \frac{T^2}{2}}{1 + T + \frac{T^2}{3}}$$

$$y_2 = \frac{\begin{bmatrix} 1 + \frac{2}{3}T & T \\ \frac{2}{3}T - 1 & T \end{bmatrix}}{\begin{bmatrix} 1 + \frac{2}{3}T & -\frac{T}{12} \\ \frac{2}{3}T - 1 & 1 + \frac{5}{12}T \end{bmatrix}} = \frac{2T}{1 + T + \frac{1}{3}T^2}$$

```
C   C  SOLUTION OF DY/DT+Y=1,TRAPEZOIDAL CONVOLUTION
       DIMENSION TIME(999),Y(999)
     1 READ,T,N
       A=T/(1.+T/2.)
       B=-(T/2.-1.)/(1.+T/2.)
       Y(1)=0.0
       PRE=0.0
       DO 2 I=2,N
       Z=I-1
       Y(I)=A+B*Y(I-1)
       X=(1.0-EXP(-T*Z))
       E=ABS(X-Y(I))
       IF(E-PRE) 2,2,8
     8 PRE=E
     2 CONTINUE
       PUNCH 11,T,PRE
    11 FORMAT(F4.3,F16.8)
       GO TO 1
       END
```

```
C   C  SOLUTION OF DY/DT+Y=1,MODIFIED SIMPSONS CONVOLUTION
       DIMENSION TIME(999),Y(999)
     1 READ,T,N
       A=1.+(5./12.)*T
       B=-((2./3.)*T-1.)
       C=(1./12.)*T
       Y(1)=0.0
       Y(2)=(T+(1./2.)*T*T)/(1.+T+(1./3.)*T*T)
       PRE=0.0
       PUNCH 10
       DO 2 I=3,N
       Z=I-1
       Y(I)=(T+B*Y(I-1)+C*Y(I-2))/A
       X=(1.0-EXP(-T*Z))
       E=ABS(X-Y(I))
       IF(E-PRE) 2,2,8
     8 PRE=E
     2 CONTINUE
       PUNCH 11,T,PRE
    10 FORMAT(//2X1HT,6X15HABS. MAX. ERROR,3X//)
    11 FORMAT(F4.3,F16.8)
       GO TO 1
       END
```

APPENDIX C

Solution of $\frac{d^2y}{dt^2} + \frac{2dy}{dt} + y = 1 \qquad \dot{y}(0) = y(0) = 0$

Laplace transforming

$$\frac{d^n}{dt^n}f(t) = s^n f(s) - \sum_{p=1}^{n} s^{n-p} f^{p-1}(0)$$

$$= s^2\bar{y} + 2s\bar{y} + \bar{y} = \frac{1}{s} \tag{84}$$

$$(s^2 + 2s + 1)\bar{y} = \frac{1}{s}$$

$$\bar{y} = \frac{1}{s(s^2+2s+1)} = \frac{1}{s(s+1)^2}$$

$$= \frac{A}{s} + \frac{B}{s+1} + \frac{C}{(s+1)^2}$$

For repeated roots

$$R(s) = \frac{P(s)}{Q(s)} (s+s_j)^r, \ \frac{P(s)}{Q(s)} = \frac{R(s)}{(s+s_j)^r} = \frac{K_{j1}}{s+sj} + \frac{K_{j2}}{(s+s_j)^2} + \cdots \frac{K_{jr}}{(s+s_j)^r}$$

$$K_{jr} = R(s)\Big|_{s=-sj} \qquad K_{jn} = \frac{1}{(r-n)!} \frac{d^{(r-n)}}{ds^{(r-n)}} R(s)\Big|_{s=-sj}$$

Now when

$$s = 0, \ A = 1$$

$$s = -1, \ C = -1$$

$$B = \frac{1}{(2-1)!} - \frac{1}{s^2}\Big|_{s=-1} = -1$$

and

$$\bar{y} = \frac{1}{4} - \frac{1}{(s+1)} - \frac{1}{(s+1)^2}$$

Inverting into the time domain

$$L\ e^{-st}t^n = \frac{n!}{(s+a)^{n+1}}$$

$$y(t) = 1-e^{-t}-e^{-t}t \quad \text{(exact solution)} \tag{85}$$

Trapezoidal Convolution:

From equation (84), dividing through by s^2

$$\bar{y} + \frac{2}{3}\bar{y} + \frac{y}{s^2} = \frac{1}{s^3} \tag{86}$$

Z-transforming (86), using equations (54), (55), and (56)

$$Z\bar{y} + \frac{T(1+z)}{(1-z)}Z\bar{y} + \frac{T^2 z}{(1-z)^2}Z\bar{y} = \frac{Tz}{(1-z)^2}\ \frac{T}{2}\frac{1+z}{1-z} = \frac{T^2}{2}\ \frac{z(1+z)}{(1-z)^3}$$

Multiplying through by $(1-z)^2$ and collecting terms

$$(T+1)Z\bar{y} + (T^2-2)zZ\bar{y} + (1-T)z^2 Z\bar{y} = T^2 X_n$$

$$\left\{X_n\right\} = \left\{0,\ \frac{1}{2},\ 1,\ 1,\ 1,\ 1,\ 1,\ \ldots\right\}$$

From which the recurrence relation is

$$(T+1)y_n + (T^2-2)y_{n-1} + (1-T)y_{n-2} = T^2 X_n \tag{87}$$

Modified Simpson's Convolution:

Z-transforming (86) using equations (55), (57), and (58)

$$Z\bar{y} + \frac{T}{(1-z)^2}\left(Tz + \frac{f_1}{12}z - \frac{f_1}{6}z^2 + \frac{f_1}{12}z^3\right)Z\bar{y} - \frac{1}{12}f_{-1}y_2 z +$$

$$\left(\frac{T}{4}y_1 + \frac{1}{6}f_{-1}y_2\right)z^2 - \frac{y_2}{12}(T+f_{-1})z^3$$

$$+ \frac{2T}{1-z}\left(\frac{5}{12} + \frac{2}{3}z - \frac{T^2}{12}\right)Z\bar{y} + \left(\frac{y_1}{4} - \frac{y_2}{12}\right)z = \frac{T^2}{2}\ \frac{z(1+z)}{(1-z)^3}$$

Multiplying through by $(1-z)^2$ and collecting terms, $(f_1=T,\ f_{-1}=-T)$

$$(1 + \tfrac{5}{6}T)Z\bar{y} + (\frac{13T^2}{12} - 2 + \frac{T}{2})zZ\bar{y} + (1 - \frac{T^2}{6} - \frac{3}{2}T)z^2Z\bar{y} + (\frac{T^2}{12} + \frac{T}{6})z^3Z\bar{y} = X_n$$

$$\left\{X_n\right\} = \left\{0,\ (\frac{T^2}{2} - \frac{T^2}{12}y_2 - \frac{T}{2}y_1 + \frac{T}{6}y_2),\ (T^2 - \frac{T^2}{4}y_1 + \frac{T^2}{6}y_2 + \frac{T}{2}y_1 - \frac{T}{6}y_2), T^2, T^2, T^2, \ldots\right\}$$

The recurrence relation is

$$(1 + \tfrac{5}{6}T)y_n + (\frac{13T^2}{12} - 2 + \frac{T}{2})y_{n-1} + (1 - \frac{T^2}{6} - \frac{3}{2}T)y_{n-2} + (\frac{T^2}{12} + \frac{T}{6})y_{n-3} = X_n \tag{88}$$

Next, y_1 and y_2 have to be determined:

From the recurrence relation

$$(1 + \tfrac{4}{3}T)y_1 + (\frac{T^2}{12} - \frac{T}{6})y_2 = \frac{T^2}{2} \tag{89}$$

$$(\tfrac{4}{3}T^2 - 2)y_1 + (1 + T - \frac{T^2}{6})y_2 = T^2 \tag{90}$$

Solving (89) and (90) for y_1 and y_2 yields

$$y_1 = \frac{\begin{bmatrix} \frac{T^2}{2} & \frac{T^2}{12} - \frac{T}{6} \\ T^2 & 1 + T - \frac{T^2}{6} \end{bmatrix}}{\begin{bmatrix} 1 + \frac{4}{3}T & \frac{T^2}{12} - \frac{T}{6} \\ \frac{4}{3}T^2 - 2 & 1 + T - \frac{T^2}{6} \end{bmatrix}} = \frac{\frac{T^2}{2} + \frac{2}{3}T^3 - \frac{T^4}{6}}{1 + 2T + \frac{4}{3}T^2 - \frac{T^4}{9}}$$

$$y_2 = \frac{\begin{bmatrix} 1 + \frac{4}{3}T & \frac{T^2}{2} \\ \frac{4}{3}T^2 - 2 & T^2 \end{bmatrix}}{\begin{bmatrix} 1 + \frac{4}{3}T & \frac{T^2}{12} - \frac{T}{6} \\ \frac{4}{3}T^2 - 2 & 1 + T - \frac{T}{6} \end{bmatrix}} = \frac{2T^2 + \frac{4}{3}T^3 - \frac{2}{3}T^4}{1 + 2T + \frac{4}{3}T^2 - \frac{T^4}{9}}$$

```
C     SOLUTION OF D(2)Y/DT(2)+2DY/DT+Y=1,TRAPEZOIDAL CONVOLUTION
      DIMENSION TIME(900),Y(900)
    1 READ,T,N
      A=T+1.
      B=-(T*T-2.)
      C=-(1.-T)
      D=T*T/2.
      Y(1)=0.0
      Y(2)=D/A
      PUNCH 10
      DO 2 I=3,N
      Z=I-1
      TIME(I)=T*Z
      Y(I)=(T*T+B*Y(I-1)+C*Y(I-2))/A
      X=(1.-EXP(-T*Z)-T*Z*EXP(-T*Z))
      E=ABS(X-Y(I))
      PCE=100.0*E/X
    2 PUNCH 11,TIME(I),X,Y(I),E,PCE
   10 FORMAT(//4HTIME,6X5HEXACT,4X7HAPPROX.,5X5HERROR,5X10HREL. ERROR//)
   11 FORMAT(F5.4,F12.8,F11.8,F11.8,F13.8)
      GO TO 1
      END
```

```
C   C  SOLUTION OF D(2)Y/DT(2)+2DY/DT+Y=1,MODIFIED SIMPSONS CONVOLUTION
       DIMENSION TIME(900),Y(900)
     1 READ,T,N
       A=1.+5./6.*T
       B=-(13./12.*T*T-2.+1./2.*T)
       C=-(1.-1./6.*T*T-3./2.*T)
       D=-(1./12.*T*T+1./6.*T)
       F=1.+2.*T+4./3.*T*T-1./9.*T*T*T*T
       G=T*T*(1./2.+(2./3.)*T-(1./6.)*T*T)
       H=2.*T*T*(1.+(2./3.)*T-(1./3.)*T*T)
       Y(1)=0.0
       Y(2)=G/F
       Y(3)=H/F
       PUNCH 10
       DO 2 I=4,N
       Z=I-1
       TIME(I)=T*Z
       Y(I)=(T*T+B*Y(I-1)+C*Y(I-2)+D*Y(I-3))/A
       X=(1.-EXP(-T*Z)-T*Z*EXP(-T*Z))
       E=ABS(X-Y(I))
       PCE=100.0*E/X
     2 PUNCH 11,TIME(I),X,Y(I),E,PCE
    10 FORMAT(//4HTIME,6X5HEXACT,4X7HAPPROX.,5X5HERROR,5X10HREL. ERROR//)
    11 FORMAT(F5.4,F12.8,F11.8,F11.8,F13.8)
       GO TO 1
       END
```

APPENDIX D

Solution of $\frac{d^3y}{dt^2} + \frac{3d^2y}{dt^2} + \frac{3dy}{dt} + y = 1 \quad y(0) = \dot{y}(0) = \ddot{y}(0) = 0$

Laplace transforming

$$s^3\bar{y} + 3s^2\bar{y} + 3s\bar{y} + \bar{y} = \frac{1}{s} \tag{91}$$

$$\bar{y} = \frac{1}{s(s^3+3s^2+3s+1)} = \frac{1}{s(s+1)^3}$$

$$= \frac{A}{s} + \frac{B}{(s+1)} + \frac{C}{(s+1)^2} + \frac{D}{(s+1)^3}$$

From which $A = 1$, $B = -2$, $C = -1$, $D = -1$

$$\bar{y} = \frac{1}{s} - \frac{2}{(s+1)} - \frac{1}{(s+1)^2} - \frac{1}{(s+1)^3}$$

Inverting into the time domain

$$y(t) = 1-e^{-t}-e^{-t}t - \frac{1}{2}e^{-t}t^2 \quad \text{(exact solution)} \tag{92}$$

Trapezoidal Convolution:

From equation (91), dividing through by s^3

$$\bar{y} + \frac{3}{s}\bar{y} + \frac{3}{s^2}\bar{y} + \frac{1}{s^3}\bar{y} = \frac{1}{s^4} \tag{93}$$

Z-transforming equation (93) using equations (55) and (56)

$$Z\bar{y} + \frac{3}{2}T \; \frac{1+z}{1-z}Z\bar{y} + \frac{3T^2z}{(1-z)^2}Z\bar{y} + \frac{T^2z}{(1-z)^2} \; \frac{T}{2}\,\frac{1+z}{1-z}Z\bar{y} = \frac{Tz}{(1-z)^2} \; \frac{T}{2}\,\frac{1+z}{1-z}^{\;2} = \frac{T^2z(1+z)^2}{4(1-z)^4}$$

Multiplying through by $(1-z)^3$ and collecting terms

$$(1 + \frac{3}{2}T)Z\bar{y} + (\frac{T^3}{2} - 3 - \frac{3}{2}T + 3T^2)zZ\bar{y} + (3 - \frac{3}{2}T + \frac{T^2}{2} - 3T^2)z^2Z\bar{y} + (\frac{3}{2}T-1)z^3Z\bar{y} = T^3X_n$$

$$\left\{X_n\right\} = \left\{0, \frac{1}{4}, \frac{3}{4}, 1, 1, 1, \ldots\right\}$$

From which the recurrence relation is

$$(1 + \frac{3}{2}T)y_n + (\frac{T^2}{2} - 3 + \frac{3}{2}T)y_{n-1} + (3 - \frac{9}{2}T + \frac{T^2}{2})y_{n-2}$$

$$+ (\frac{3}{2}T - 1)y_{n-3} = T^3X_n \qquad (94)$$

Modified Simpson's Convolution:

Z-transforming (93) using equations (55), (57), (58) and (59) yields

$$Z\bar{y} + \frac{3T}{1-z}\ (\frac{5}{12} + \frac{2}{3}z - \frac{z^2}{12})Z\bar{y} + (\frac{y_1}{4} - \frac{y_2}{12})z$$

$$+ \frac{3T}{(1-z)^2}\ (Tz + \frac{Tz}{12} - \frac{Tz^2}{6} + \frac{Tz^3}{12})Z\bar{y} + \frac{y_2}{12}z$$

$$+ (\frac{T}{4}y_1 - \frac{T}{6}y_2)z^2$$

$$+ \frac{T}{2(1-z)^3}\ (T^2z + T^2z^2 + \frac{T^2}{12}z - \frac{T^2}{4}z^2 + \frac{T^2}{4}z^3 - \frac{T^2}{12}z^4)Z\bar{y}$$

$$- \frac{T^2}{12}y_2z + (\frac{T^2}{4}y_1 + \frac{T^2}{4}y_2)z^2$$

$$+ (\frac{T^2}{4}y_1 - \frac{T^2}{4}y_2 - \frac{T^2}{12}y_2)z^3 = \frac{T^3z(1+z)^2}{4(1-z)^4}$$

Multiplying through by $(1-z)^3$ and collecting terms, the recurrence relation is

$$(1 + \frac{5}{4}T)y_n + (\frac{13T^2}{4} - \frac{T}{2} - 3 + \frac{13}{24}T^3)y_{n-1} + (3 - 3T - \frac{15}{4}T^2 + \frac{3}{8}T^3)y_{n-2}$$

$$+ (\frac{5}{2}T - 1 + \frac{3}{4}T^2 + \frac{T^3}{8})y_{n-3} - (\frac{T}{4} + \frac{T^2}{4} + \frac{T^3}{24})y_{n-4} = X_n \qquad (95)$$

$$\left\{X_n\right\} = \left\{0, (\frac{T}{4}y_2 - \frac{T^2}{4}y_2 + \frac{T^3}{4} - \frac{3T}{4}y_1 + \frac{T^3}{24}y_2), (\frac{3}{4}T^3 + \frac{3}{4}T^2y_2 - \frac{T}{2}y_2 + \frac{3T}{2}y_1 - \right.$$

$$\frac{3}{4}T^2y_1 - \frac{T^3}{8}y_1 - \frac{T^3}{8}y_2), (T^3 - \frac{3T}{4}y_1 + \frac{3T^2}{4}y_1 - \frac{T^3}{8}y_1 + \frac{T}{4}y_2 - \frac{T^2}{2}y_2 + \frac{T^3}{6}y_2),$$

$$\left. T^3, T^3, T^3, \ldots \right\}$$

Next, y_1 and y_2 have to be determined. From the recurrence relation

$$(1 + 2T)y_1 - (\frac{T^3}{24} - \frac{T}{4} + \frac{T^2}{4})y_2 = \frac{T^3}{4} \tag{96}$$

$$(4T^2 - 2T - 3 + \frac{2T^3}{3})y_1 + (1 + \frac{2T}{4} + \frac{T^3}{8} - \frac{3T^2}{4}) = \frac{3}{4}T^3 \tag{97}$$

Solving equations (96) and (97) for y_1 and y_2 yields

$$y_1 = \frac{\begin{bmatrix} \frac{T^3}{4} & -\frac{T^3}{24} + \frac{T}{4} - \frac{T^2}{4} \\ \frac{3T^3}{4} & 1 + \frac{7T}{4} + \frac{T^3}{8} - \frac{3T^2}{4} \end{bmatrix}}{\begin{bmatrix} 1+2T & -\frac{T^3}{24} + \frac{T}{4} - \frac{T^2}{4} \\ 4T^2-2T-3+\frac{2T^3}{3} & 1 + \frac{7T}{4} + \frac{T^3}{8} - \frac{3}{4}T^2 \end{bmatrix}} = \frac{\frac{T^3}{4} + \frac{T^4}{4} + \frac{T^6}{16}}{1 + \frac{9}{2}T + \frac{5T^2}{2} - 3T^3 + T^4 + \frac{T^5}{3} + \frac{T^6}{36}}$$

$$y_2 = \frac{\begin{bmatrix} 1+2T & \frac{T^3}{4} \\ 4T^2-2T-3+\frac{2}{3}T^3 & \frac{3}{4}T^3 \end{bmatrix}}{\begin{bmatrix} 1+2T & -\frac{T^3}{24} + \frac{T}{4} - \frac{T^2}{4} \\ 4T^2-2T-3+\frac{2}{3}T^3 & 1 + \frac{7T}{4} + \frac{T^3}{8} - \frac{3T^2}{4} \end{bmatrix}} = \frac{\frac{3T^3}{2} + \frac{7}{4}T^4 - T^5 - \frac{T^6}{3}}{1 + \frac{9}{2}T + \frac{5T^2}{2} - 3T^3 + T^4 + \frac{T^5}{3} + \frac{T^6}{36}}$$

```
C   C   SOLUTION OF D(3)Y/DT(3)+3D(2)Y/DT(2)+3DY/DT+Y=1,TRAPEZOIDAL CONV.
        DIMENSION TIME(900),Y(900)
      1 READ,T,N
        A=1.+(3./2.)*T
        B=-(T*T*T/2.-3.-3./2.*T+3.*T*T)
        C=-(3.-3./2.*T-3.*T*T+T*T*T/2.)
        D=-((3./2.)*T-1.)
        F=T*T*T/4.
        G=(3./4.)*T*T*T
        Y(1)=0.0
        Y(2)=F/A
        Y(3)=(G+B*Y(2))/A
        PUNCH 10
        DO 2 I=4,N
        Z=I-1
        TIME(I)=T*Z
        Y(I)=(T*T*T+B*Y(I-1)+C*Y(I-2)+D*Y(I-3))/A
        X=1.-1.*EXP(-T*Z)-T*Z*EXP(-T*Z)-(1./2.)*T*T*Z*Z*EXP(-T*Z)
        E=ABS(X-Y(I))
        IF(X) 3,5,3
      3 PCE=100.*E/X
        IF(ABS(PCE)-1000.) 4,5,5
      4 PUNCH 11,TIME(I),X,Y(I),E,PCE
      2 CONTINUE
     10 FORMAT(//4HTIME,6X5HEXACT,4X7HAPPROX.,5X5HERROR,5X10HREL. ERROR//)
     11 FORMAT(F5.4,F12.8,F11.8,F11.8,F13.8)
        GO TO 1
      5 PUNCH 11,TIME(I),X,Y(I),E
        GO TO 2
        END
```

```
C   C  SOLUTION OF D(3)Y/DT(3)+3D(2)Y/DT(2)+3DY/DT+Y=1,MOD. SIMP. CONV.
       DIMENSION TIME(900),Y(900)
     1 READ,T,N
       A=1.+(5./4.)*T
       B=-((13./4.)*T*T-(1./2.)*T-3.+(13./24.)*T*T*T)
       C=-(3.-3.*T-(15./4.)*T*T+(3./8.)*T*T*T)
       D=-((5./2.)*T-1.+(3./4.)*T*T+(1./8.)*T*T*T)
       F=(1./4.)*T+(1./4.)*T*T+(1./24.)*T*T*T
       G=1.+9./2.*T+5./2.*T*T-3.*T*T*T*(1.-T/3.-1./9.*T*T-1./108.*T*T*T)
       H=T*T*T*(1./4.+(1./4.)*T+(1./16.)*T*T*T)
       O=T*T*T*(3./2.+7./2.*T-T*T-(1./3.)*T*T*T)
       Y(1)=0.0
       Y(2)=H/G
       Y(3)=O/G
       P=(T*T*T+(3./4.)*T*Y(2)*(-1.+T-(1./6.)*T*T))
       Q=(I./4.)*T*Y(3)*(1.-2.*T+2./3.*T*T)
       R=P+Q
       Y(4)=(R+B*Y(3)+C*Y(2))/A
       Y(5)=(T*T*T+B*Y(4)+C*Y(3)+D*Y(2))/A
       PUNCH 10
       DO 2 I=5,N
       Z=I-1
       TIME(I)=T*Z
       Y(I)=(T*T*T+B*Y(I-1)+C*Y(I-2)+D*Y(I-3)+F*Y(I-4))/A
       X=1.-1.*EXP(-T*Z)-T*Z*EXP(-T*Z)-(1./2.)*T*T*Z*Z*EXP(-T*Z)
       E=ABS(X-Y(I))
       IF(X) 3,5,3
     3 PCE=100.*E/X
       IF(ABS(PCE)-1000.) 4,5,5
     4 PUNCH 11,TIME(I),X,Y(I),E,PCE
     2 CONTINUE
    10 FORMAT(//4HTIME,6X5HEXACT,4X7HAPPROX.,5X5HERROR,5X10HREL. ERROR//)
    11 FORMAT(F5.4,F12.8,F11.8,F11.8,F13.8)
       GO TO 1
     5 PUNCH 11,TIME(I),X,Y(I),E
       GO TO 2
       END
```

APPENDIX E

Solution of $\frac{dy}{dt} + y^2 = 1 \quad y(0) = 0$

The known exact solution is

$$y(t) = \tanh t = \frac{e^t - e^{-t}}{e^t + e^{-t}} = 1 - \frac{2}{1 + e^{2t}} \tag{98}$$

Trapezoidal Convolution:

Laplace transforming

$$s\bar{y} + L\left[y^2\right] = \frac{1}{s} \tag{99}$$

Dividing (99) through by s

$$\bar{y} = \frac{1}{s}L\left[y^2\right] = \frac{1}{s^2} \tag{100}$$

Z-transforming equation (100) using equations (54) and (55)

$$Z\bar{y} + \frac{T}{2}\frac{1+z}{1-z}\,Z\left[y^2\right] = \frac{Tz}{(1-z)^2}$$

Multiplying through by (1-z)

$$Z\bar{y}(1-z) + \frac{T}{2}(1+z)\,Z\left[y^2\right] = \frac{Tz}{1-z}$$

From which the recurrence relation is

$$y_n - y_{n-1} + \frac{T}{2}y_n^2 + \frac{T}{2}y_{n-1}^2 = TX_n \qquad \left\{X_n\right\} = \left\{0,\ 1,\ 1,\ 1,\ 1,\ \dots\right\} \tag{101}$$

Modified Simpson's Convolution:

Z-transforming equation (100) using equations (55) and (57)

$$Z\bar{y} + \frac{T}{1-z}\left(\frac{5}{12} + \frac{2}{3}z - \frac{z^2}{12}\right) Z\,y^2 + \left(\frac{y_1}{4} - \frac{y_2}{12}\right)z = \frac{Tz}{(1-z)^2}$$

Multiplying through by (1-z) and collecting terms

$$z\bar{y}(1-z) + (\frac{5}{12}T + \frac{2}{3}Tz - \frac{T}{12}z^2)\ z\ y^2 = \frac{Tz}{(1-z)} + (\frac{T}{12}y_2 - \frac{T}{4}y_1)$$

From which the recurrence relation is

$$y_n - y_{n-1} + \frac{5}{12}Ty_n^2 + \frac{2}{3}Ty_{n-1}^2 - \frac{T}{12}y_{n-2}^2 = X_n \qquad (102)$$

$$\left\{X_n\right\} = \left\{0, (T + \frac{T}{12}y_2 - \frac{T}{4}y_1),\ T,\ T,\ \ldots\right\}$$

Next, y_1 and y_2 have to be determined.

These starting conditions cannot be calculated as previously calculated, therefore y_1 and y_2 will be found by applying a series solution to the original differential equation, $dy/dt + y^2 = 1$, (7).

$$\frac{dy}{dt} = 1 - y^2 \qquad y(0) = 0 \qquad (103)$$

assume a solution

$$y = c_1t + c_2t^2 + c_3t^3 \ldots \qquad (104)$$

$$y^1 = c_1 + 2c_2t + 3c_3t^2 \ldots \qquad (105)$$

Substituting equations (104) and (105) into equation (103) yields

$$c_1 + 2c_2t + 3c_3t^2 + \ldots = 1 - \left[c_1t + c_2t^2 + c_3t^3 \ldots\right]^2$$

$$= 1 - c_1^2t^2 - 2c_1c_2t^3 - (c_2^2 + 2c_1c_3)t^4 \qquad (106)$$

$$- (2c_1c_4 + 2c_2c_3)t^5 - (c_3^2 + 2c_1c_5 + 2c_2c_4)t^6 + \ldots$$

Equating like power of t in equation (106) yields

$c_1 = 1$, $c_2 = 0$, $c_3 = -\frac{1}{3}$, $c_4 = 0$, $c_5 = \frac{2}{15}$, $c_6 = 0$, $c_7 = \frac{5.67}{105}$

Therefore

$$y = t - \frac{t^3}{3} + \frac{2}{15}t^5 - \frac{5.67}{105}t^7 \ldots$$

$$y_n = nT - \frac{(nT)^3}{3} - \frac{2(nT)^5}{15} - \frac{5.67(nT)^7}{105} \ldots \qquad (107)$$

Equation (107) yields

$$y_1 = T - \frac{T^3}{3} + \frac{2}{15}T^5 - \frac{5.67}{105}T^7 \qquad (108)$$

$$y_2 = 2T - \frac{8}{3}T^3 + \frac{64}{15}T^5 - \frac{726}{105}T^7 \qquad (109)$$

```
C   C  SOLUTION OF DY/DT+Y*Y=1,TRAPEZOIDAL CONVOLUTION
       DIMENSION TIME(900),Y(900)
     1 READ,T,N
       A=-1./T
       B=2.+1./(T*T)
       C=2./T
       Y(1)=0.0
       PUNCH 10
       DO 2 I=2,N
       Z=I-1
       TIME(I)=T*Z
       Y(I)=A+SQRT(B+C*Y(I-1)-Y(I-1)*Y(I-1))
       X=1.-2./(1.+EXP(2.*T*Z))
       E=ABS(X-Y(I))
       PCE=100.0*E/X
     2 PUNCH 11,TIME(I),X,Y(I),E,PCE
    10 FORMAT(//4HTIME,6X5HEXACT,4X7HAPPROX.,5X5HERROR,5X10HREL. ERROR//)
    11 FORMAT(F5.3,F12.8,F11.8,F11.8,F13.8)
       GO TO 1
       END
```

```
C  C  SOLUTION OF DY/DT+Y*Y=1,MODIFIED SIMPSONS CONVOLUTION
      DIMENSION TIME(900),Y(900)
    1 READ,T,N
      A=-(6./5.)*1./T
      B=(36./25.)*(1./(T*T))+12./5.
      C=1./5.
      Y(1)=0.0
      Y(2)=T-T*T*T*(1./3.)*(1.-2./5.*T*T+(5.67/35.)*T*T*T*T)
      PUNCH 10
      DO 2 I=3,N
      Z=I-1
      TIME(I)=T*Z
      Y(I)=A+SQRT(B+C*(12.*(1./T)*Y(I-1)-8.*Y(I-1)**2+Y(I-2)**2))
      X=1.-2./(1.+EXP(2.*T*Z))
      E=ABS(X-Y(I))
      PCE=100.0*E/X
    2 PUNCH 11,TIME(I),X,Y(I),E,PCE
   10 FORMAT(//4HTIME,6X5HEXACT,4X7HAPPROX.,5X5HERROR,5X10HREL. ERROR//)
      GO TO 1
      END
```

INVESTIGATION OF NUMERICAL INTEGRATION TECHNIQUES
WITH APPLICATIONS TO THE NUMERICAL SOLUTION
OF DIFFERENTIAL EQUATIONS

by

LARRY DAN FOSTER

B.S.M.E., Kansas State University, 1960
B.S.E.E., Kansas State University, 1963

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The essential purpose of this report has been to develop a Z-transform notation for Simpson's rule similar to that of Trapezoidal Convolution and use this development in the solution of some differential equations.

Generally speaking, the Z-transform notation must be dropped when using Simpson's formula because of the finite number of starting values that are required. This makes the finite difference equation ultimately recursive, while Trapezoidal Convolution requires only one starting value, making it instantly recursive. A method was found to overcome this fault and Z-transforms were effective in carrying out numerical examples using the Modified Simpson's Convolution.

In trying to find an approximation of $Z\left[1/s^n\right]$ in a similar manner as that used for Trapezoidal Convolution, negative results were arrived at and Criswell's exact $Z\left[1/s^n\right]$ was used in the Modified Simpson's Convolution formula. With this final touch, the integrator substitution program was developed and described in Z-transform notation.

An empirical error analysis of a first order differential equation was made to determine the maximum absolute error for various sampling intervals, for a definite solution time. The error analysis shows that there exists an optimum sampling range which minimizes the largest absolute value of error using Trapezoidal Convolution and the Modified Simpson's Convolution; that Trapezoidal Convolution has the best stability with poorer results; that the Modified Simpson's Convolution has better results and its stability is much improved over Simpson's 1/3 rule.

When using higher-order Newton-Cotes quadratures such as the Modified Simpson's Convolution, the estimation of starting points plays a vital role. The better the estimate the higher the accuracy in numerical analysis.

Comparing decimal place accuracy between the Modified Simpson's Convolution and Trapezoidal Convolution for various sampling intervals it was observed that Trapezoidal Convolution yielded overall better accuracy and required less computer time; the algebra in determining the starting conditions for the Modified Simpson's Convolution is long and tedious for the second and third order differential equations solved; both methods dropped in accuracy when solving the non-linear equation which required a square root operation, and that the starting conditions for the non-linear equation had to be determined by using a Taylor series or a differential equation series instead of using determinants which were used for the other differential equations solved.

In conclusion, if one is looking for stability and minimum absolute maximum error, the Modified Simpson's Convolution may suffice, but if one is looking for decimal place accuracy, Trapezoidal Convolution is advantageous up to and including the third order differential equation solved.